A Solution Manual for the Problems in Daniel Baumann's "*Cosmology*" [1]

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Part I

The Homogeneous Universe

Chapter 1

Introduction

1.1

1. $$\frac{V_{\text{Earth}}}{V_{\text{Moon}}} \approx 50. \tag{1.1}$$

   Thus, if the Earth were scaled to the size of a basketball, the Moon would correspond roughly to a tennis ball.

   $$\frac{r_{\text{Moon-Earth}}}{r_{\text{Earth}}} \approx 60. \tag{1.2}$$

   The Earth–Moon distance would be comparable to the distance from the three-point line to the basket.

2. Taking a peppercorn of diameter $d_{\text{peppercorn}} \sim 5$ mm to represent Earth, the scaled Sun–Earth distance (1 AU) becomes

   $$r_{\text{scaled Sun-Earth}} = \frac{d_{\text{peppercorn}}}{d_{\text{Earth}}} \times r_{\text{Sun-Earth}} \sim \frac{5\text{ mm}}{12742.018\text{ km}} \times 1\text{ AU} \approx \boxed{60\text{ m}}\,. \tag{1.3}$$

   The average orbital radius of Neptune is $D_{\text{Sun-Neptune}} \approx 30.1$ AU, which under the same scaling becomes

   $$r_{\text{scaled Sun-Neptune}} = \frac{d_{\text{peppercorn}}}{d_{\text{Earth}}} \times r_{\text{Sun-Neptune}} \sim \frac{5\text{ mm}}{12742.018\text{ km}} \times 30.2\text{ AU} \approx \boxed{1800\text{ m}}\,. \tag{1.4}$$

3. We take the average orbital radius of Neptune as a proxy for the Solar System's radius.[1] For reference, a standard international basketball court measures 28 m × 15 m. Under this scaling, the Solar System would shrink to

   $$d_{\text{scaled Solar System}} = \frac{d_{\text{Solar System}}}{d_{\text{Solar Neighborhood}}} \times l_{\text{basketball court}} \approx \frac{60.4\text{ AU}}{65\text{ ly}} \times 28\text{ m} \approx \boxed{4.1 \times 10^{-4}\text{ m}}\,. \tag{1.5}$$

[1] This, of course, strongly underestimates the size of the Solar System, and its boundary is not even well defined. Depending on whether one considers the Oort Cloud, heliopause, heliosphere, or Kuiper Belt, the result would vary significantly.

4.

$$d_{\text{scaled Solar Neighborhood}} = \frac{d_{\text{Solar Neighborhood}}}{d_{\text{Milky Way}}} \times l_{\text{basketball court}} \approx \frac{65 \text{ lys}}{10^5 \text{ lys}} \times 28 \text{ m} \approx \boxed{0.018 \text{ m}} . \tag{1.6}$$

5.

$$d_{\text{scaled Milky Way}} = \frac{d_{\text{Milky Way}}}{d_{\text{Local Group}}} \times l_{\text{basketball court}} \approx \frac{10^5 \text{ lys}}{10^7 \text{ lys}} \times 28 \text{ m} \approx \boxed{0.28 \text{ m}} . \tag{1.7}$$

6.

$$d_{\text{scaled Local Group}} = \frac{d_{\text{Local Group}}}{d_{\text{Local Supercluster}}} \times l_{\text{basketball court}} \approx \frac{10^7 \text{ lys}}{5 \times 10^8 \text{ lys}} \times 28 \text{ m} \approx \boxed{0.56 \text{ m}} . \tag{1.8}$$

7.

$$d_{\text{scaled Local Supercluster}} = \frac{d_{\text{Local Supercluster}}}{d_{\text{observable universe}}} \times l_{\text{basketball court}} \approx \frac{5 \times 10^8 \text{ lys}}{2 \times 46.5 \times 10^9 \text{ lys}} \times 28 \text{ m} \approx \boxed{0.15 \text{ m}} . \tag{1.9}$$

1.2

1.

$$t_{H_0} \equiv H_0^{-1} = 70^{-1} \text{ km}^{-1} \text{ s Mpc} \approx \boxed{4.4 \times 10^{17} \text{ s}} . \tag{1.10}$$

2.

$$d_{H_0} \equiv cH_0^{-1} = ct_{H_0} \approx \boxed{1.3 \times 10^{26} \text{ m}} . \tag{1.11}$$

Because the universe is expanding, this is not the physical radius of the observable universe, but only about one-third of it.

3.

$$\rho_0 = \frac{3H_0^2}{8\pi G} \approx \boxed{9.2 \times 10^{-27} \text{ kg m}^{-3}} . \tag{1.12}$$

This is roughly $\boxed{10^{-29}}$ times smaller than that of water, $\rho_w \approx 1.0 \times 10^3 \text{ kg m}^{-3}$.

4.

$$N_{H,universe} = \frac{m_{\text{universe}}}{m_H} = \frac{\rho_0 V_{\text{universe}}}{m_H} \sim \rho_0 \frac{4}{3}\pi d_{H_0}^3 \frac{1}{m_H} \approx \boxed{10^{80}} . \tag{1.13}$$

The molar mass of hydrogen is 1 g/mol, and that of oxygen is 16 g/mol. Thus, the hydrogen mass fraction in water is $\frac{2\times1 \text{ g/mol}}{16 \text{ g/mol}+2\times1 \text{ g/mol}} = \frac{1}{9}$. Taking the average adult brain mass $m_{\text{brain}} \approx 1.4$ kg and assuming it consists mostly of water,

$$N_{H,brain} = \frac{m_{\text{brain}} \times \frac{1}{9}}{m_H} \approx 10^{26}, \tag{1.14}$$

which is only $\boxed{10^{-54}}$ of the total hydrogen atoms in the universe.

5.

$$l_{\min} = \frac{\hbar c}{E_{\max}} \approx \boxed{2.0 \times 10^{-19}\ \mathrm{m}}\ . \tag{1.15}$$

This scale is about $\boxed{10^{-45}}$ times smaller than the Hubble distance d_{H_0}.

Chapter 2

The Expanding Universe

Exercise 2.1

The Euler-Lagrange equation tells us

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}, \tag{2.1}$$

where q is a generalized coordinate. Applying to the Lagrangian of the free particle

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2\right), \tag{2.2}$$

gives us

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{r}}\right) &= \frac{\partial L}{\partial r} \\ \ddot{r} &= r\dot{\phi}^2, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\phi}}\right) &= \frac{\partial L}{\partial \phi} \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(mr^2\dot{\phi}\right) &= 0 \\ \ddot{\phi} &= -\frac{2}{r}\dot{r}\dot{\phi}. \end{aligned} \tag{2.4}$$

Exercise 2.2

From the FRW metric,

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a^2(t)\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j. \tag{2.5}$$

Since $g_{00} = -c$ is a constant and $g_{i0} = g_{0i} = 0$, any Christoffel symbols with at least two time indices vanish, i.e. $\Gamma^{\mu}_{00} = \Gamma^{0}_{0\beta} = 0$.

$$\begin{aligned}
\Gamma^0_{ij} &= \frac{1}{2} g^{0\lambda} \left(\partial_i g_{j\lambda} + \partial_j g_{i\lambda} - \partial_\lambda g_{ij} \right) \\
&= \frac{1}{2} g^{00} \left(\partial_i g_{j0} + \partial_j g_{i0} - \partial_0 g_{ij} \right) \\
&= \frac{1}{2} \partial_0 g_{ij} \\
&= \frac{1}{2} \partial_0 \left(a^2 \right) \gamma_{ij} \\
&= c^{-1} a \dot{a} \gamma_{ij} .
\end{aligned} \tag{2.6}$$

$$\begin{aligned}
\Gamma^i_{0j} &= \frac{1}{2} g^{i\lambda} \left(\partial_0 g_{j\lambda} + \partial_j g_{0\lambda} - \partial_\lambda g_{0j} \right) \\
&= \frac{1}{2} g^{ik} \left(\partial_0 g_{jk} \right) \\
&= \frac{1}{2} a^{-2} \gamma^{ik} \partial_0 \left(a^2 \right) \gamma_{jk} \\
&= c^{-1} \frac{\dot{a}}{a} \delta^i_j ,
\end{aligned} \tag{2.7}$$

where we have used the fact that the metric is symmetric such that $\gamma_{ij} = \gamma_{ji}$.

$$\begin{aligned}
\Gamma^i_{jk} &= \frac{1}{2} g^{i\lambda} \left(\partial_j g_{k\lambda} + \partial_k g_{j\lambda} - \partial_\lambda g_{jk} \right) \\
&= \frac{1}{2} g^{il} \left(\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk} \right) \\
&= \frac{1}{2} \gamma^{il} \left(\partial_j \gamma_{kl} + \partial_k \gamma_{jl} - \partial_l \gamma_{jk} \right) .
\end{aligned} \tag{2.8}$$

Exercise 2.3

Starting from Eq. (2.55) of the textbook,

$$\begin{aligned}
g_{\mu\nu} P^\mu P^\nu &= -m^2 c^2 \\
g_{00} P^0 P^0 + g_{ij} P^i P^j &= -m^2 c^2 \\
-(P^0)^2 + p^2 &= -m^2 c^2 .
\end{aligned} \tag{2.9}$$

Taking time derivative on both sides:

$$-P^0 \frac{\mathrm{d}P^0}{\mathrm{d}t} + p \frac{\mathrm{d}p}{\mathrm{d}t} = 0. \tag{2.10}$$

Now, using the geodesic equation for P^0 (Eq. (2.52) of the textbook), we shall get

$$p \frac{\mathrm{d}p}{\mathrm{d}t} = P^0 \frac{\mathrm{d}P^0}{\mathrm{d}t} = \frac{E}{c^2} \frac{\mathrm{d}E}{\mathrm{d}t} = -a \dot{a} \gamma_{ij} P^i P^j = -\dot{a} p^2 , \tag{2.11}$$

or

$$\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\dot{a}}{a}. \tag{2.12}$$

Thus, $p \propto a^{-1}$.

To derive Eq. (2.56) of the textbook, we notice that from the metric,

$$c^2\mathrm{d}\tau^2 = c^2\mathrm{d}t^2 - g_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{2.13}$$

or

$$\left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)^2 = 1 - \frac{v^2}{c^2}. \tag{2.14}$$

Then, since

$$p^2 \equiv g_{ij}P^iP^j = g_{ij}m^2\frac{\mathrm{d}x^i}{\mathrm{d}\tau}\frac{\mathrm{d}x^j}{\mathrm{d}\tau} = g_{ij}m^2\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t} = m^2v^2\left(1 - \frac{v^2}{c^2}\right)^{-1}, \tag{2.15}$$

we have

$$p = \frac{mv}{\sqrt{1 - v^2/c^2}}. \tag{2.16}$$

Exercise 2.4

From $U = (\rho c^2)V$, we know that

$$-P\mathrm{d}V = \mathrm{d}U = \mathrm{d}\rho c^2 V + \rho c^2 \mathrm{d}V \tag{2.17}$$

Dividing both sides by $\mathrm{d}t$,

$$-P\dot{V} = \dot{\rho}c^2V + \rho c^2\dot{V}, \tag{2.18}$$

or

$$\dot{\rho} + \frac{\dot{V}}{V}\left(\rho + \frac{P}{c^2}\right) = 0. \tag{2.19}$$

On the other hand, we know that since $V \propto a^3$,

$$\frac{\dot{V}}{V} = \frac{1}{a^3}\frac{\mathrm{d}(a^3)}{\mathrm{d}t} = 3\frac{\dot{a}}{a}. \tag{2.20}$$

Plugging this back into Eq. (2.19), we arrive at

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) = 0. \tag{2.21}$$

Exercise 2.5

- First noticing that since $\rho_m a^3 = \frac{1}{2}\rho_{\mathrm{eq}}a_{\mathrm{eq}}^3$ is a constant, and similarly $\rho_r a^4 = \frac{1}{2}\rho_{\mathrm{eq}}a_{\mathrm{eq}}^4$, we can infer that nowadays ($a = 1$) we have $\Omega_m a_{\mathrm{eq}} = \Omega_m y_0^{-1} = \Omega_r$, where we have defined $y_0 \equiv y(z = 0) = 1 + z_{\mathrm{eq}} \equiv a_{\mathrm{eq}}^{-1}$.

Starting from the Eq. (2.194) of the textbook with $\Omega_\Lambda = \Omega_k = 0$, using $y \equiv \frac{1+z_{\rm eq}}{1+z} = \frac{a}{a_{\rm eq}} = y_0 a$ or $\mathrm{d}a = y_0^{-1}\mathrm{d}y$, we have

$$\begin{aligned} t &= \frac{1}{H_0}\int_0^a \frac{\mathrm{d}a'}{\sqrt{\Omega_r a'^{-2} + \Omega_m a'^{-1}}} \\ &= \frac{1}{H_0}\int_0^y \frac{y'\mathrm{d}y'}{y_0^{3/2}\sqrt{\Omega_m}\sqrt{1+y'}} \\ &= \frac{1}{H_0 y_0^{3/2}\sqrt{\Omega_m}}\int_0^y \mathrm{d}y'\left[\sqrt{1+y'} - \frac{1}{\sqrt{1+y'}}\right] \\ &= \frac{1}{H_0 y_0^{3/2}\sqrt{\Omega_m}}\left[\frac{4}{3} + \frac{2}{3}(1+y)^{3/2} - 2(1+y)^{1/2}\right]. \end{aligned} \tag{2.22}$$

Plugging the observed values $H_0 = 67.74~\mathrm{km\,s^{-1}\,Mpc^{-1}}$, $y_0 = 1 + z_{\rm eq} = 3401$, and $\Omega_m = 0.3153$, we can calculate the prefactor to be ≈ 130000 yrs, and thus

$$t = 130000 \text{ yrs}\left[\frac{4}{3} + \frac{2}{3}(1+y)^{3/2} - 2(1+y)^{1/2}\right]. \tag{2.23}$$

- At $z = z_{\rm eq}$, $y = 1$, and thus, the matter-radiation equality happens at $t \approx 50000$ yrs.
- At $z = z_{\rm rec} = 1100$, $y = \frac{3401}{1101}$, and thus, the recombination happens at $t \approx 360000$ yrs.

• Similarly, setting instead $\Omega_r = \Omega_k = 0$, $\Omega_m = a_{m\Lambda}^3\Omega_\Lambda = y_0^{-3}\Omega_\Lambda$ and starting from the Eq. (2.194) of the textbook again, we get

$$\begin{aligned} t &= \frac{1}{H_0}\int_0^a \frac{\mathrm{d}a'}{\sqrt{\Omega_m a'^{-1} + \Omega_\Lambda a'^2}} \\ &= \frac{1}{H_0\sqrt{\Omega_\Lambda}}\int_0^y \frac{\sqrt{y'}\mathrm{d}y'}{\sqrt{1+y'^3}} \\ &= \frac{2}{3H_0\sqrt{\Omega_\Lambda}}\sinh^{-1}(y^{3/2}). \end{aligned} \tag{2.24}$$

Plugging the observed values $H_0 = 67.74~\mathrm{km\,s^{-1}\,Mpc^{-1}}$, $y_0 = 1 + z_{m\Lambda} = 1.3$, and $\Omega_\Lambda = 0.6847$ give us the prefactor to be ≈ 11.5 Gyrs. Thus, we have

$$t = 11.5 \text{ Gyrs} \times \sinh^{-1}(y^{3/2}). \tag{2.25}$$

- At $z = z_{m\Lambda}$, $y = 1$, and thus, the matter-dark energy equality happens at $t \approx 10.1$ Gyrs.
- At $z = z_0 = 0$, $y = 1 + z_{m\Lambda} = 1.3$, and thus, the age of the universe is about $t \approx 13.6$ Gyrs.

2.1 Robertson-Walker metric

1. • **Why $g_{00} = -1$?**

Homogeneity implies that g_{00} can only be a function of time with no space dependence. Now suppose one has a non-trivial g_{00}, one can always do a redefinition of the time coordinate $\mathrm{d}t' \equiv \sqrt{-g_{00}}\mathrm{d}t$ to absorb the non-trivial g_{00} so one can always rescale to set $g_{00} = -1$.

- **Why $\boldsymbol{g_{0i} = 0}$?**
  Isotropy requires the mean values of any three-vectors to vanish in the comoving frame for there is no preferred direction, and thus one must have $g_{0i} = 0$.
- **Why $\boldsymbol{g_{ij} = a^2(t)\gamma_{ij}(\vec{x})}$?**
  Isotropy around a point $\mathbf{x} = 0$ constrains the mean values of any three-tensor to be proportional to δ_{ij}. Homogeneity requires the proportionality coefficient to be only a function of time. Also, as the proportionality is unaffected by transformation of the spatial coordinates, one can always separate the g_{ij} into a time-dependent part and a spatial-dependent part. Since there is nothing special about the point $\mathbf{x} = 0$ by homogeneity, the factorization must hold everywhere.

2. • Isotropy implies a rotational invariance, under which the radial component r is preserved under a rotation transformation. This implies g_{rr} can only be a function of radial coordinate alone: $g_{rr} = A(r)$. As it never runs to negative value by definition, one can choose

$$g_{rr} = A(r) \equiv e^{2\alpha(r/R_0)} \tag{2.26}$$

for some function $\alpha(r/R_0)$. The R_0 is just some arbitrary constant carrying the a dimension of [length] to make the metric dimensionally consistent.

Also, there can never be non-vanishing mixing terms between the radial components and angular components such as $g_{r\theta}$ and $g_{r\phi}$. The rotational invariance also implies $g_{r\theta}$ and $g_{r\phi}$ can only be functions of radial component r only. However, if one does a reflection redefinition on the angular angles (e.g., $\theta \to \pi - \theta$ or $\phi \to -\phi$), which leaves r unchanged, but $\mathrm{d}r\mathrm{d}\theta$ or $\mathrm{d}r\mathrm{d}\phi$ picks a sign change under such transformation, and one concludes

$$g_{r\theta}(r)\mathrm{d}r\mathrm{d}\theta = -g_{r\theta}(r)\mathrm{d}r\mathrm{d}\theta = 0, \tag{2.27}$$

and

$$g_{r\phi}(r)\mathrm{d}r\mathrm{d}\phi = -g_{r\phi}(r)\mathrm{d}r\mathrm{d}\phi = 0. \tag{2.28}$$

Moreover, the rotational invariance also implies the angular coordinates mix in such a way that the standard metric on the sphere $\mathrm{d}\Omega^2 \equiv \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2$ is preserved with a prefactor that can only be a function of radial component r only. More specifically,

$$g_{\theta\theta}(r,\theta,\phi)\mathrm{d}\theta^2 + 2g_{\theta\phi}(r,\theta,\phi)\mathrm{d}\theta\mathrm{d}\phi + g_{\phi\phi}(r,\theta,\phi)\mathrm{d}\phi^2 = B(r)\mathrm{d}\Omega^2. \tag{2.29}$$

Now, the rotational invariance also says the metric should not change under a linear shift on the azimuthal angle $\phi \to \phi + \alpha$ for an arbitrary constant α. This immediately implies $g_{\theta\theta}$, $g_{\theta\phi}$, and $g_{\phi\phi}$ can not have any ϕ dependence. A reflection arguments again implies the cross term should vanish:

$$g_{\theta\phi}(r,\theta)\mathrm{d}\theta\mathrm{d}\phi = -g_{\theta\phi}(r,\theta)\mathrm{d}\theta\mathrm{d}\phi = 0. \tag{2.30}$$

Finally, for any fixed r, the angular part of the metric needs to be isomorphic to a 2-sphere with radius r. This fixes $B(r) = r^2$ or otherwise, one can not reproduce the correct physical circumference. Therefore, the most general spatial metric takes the form

$$\mathrm{d}\ell^2 \equiv e^{2\alpha(r/R_0)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2. \tag{2.31}$$

- Notice that we're calculating a 3-scalar, which is invariant under any spacial transformations. Thus, we can choose the coordinate with $\theta = \frac{\pi}{2}$, which shall significantly simplify the calculation. Also, since the metric has no off-diagonal elements, any Christoffel symbols that has the form Γ^i_{jk} with $i \neq j \neq k$ must vanish. In other words, there must be at least two repeated indices. The non-vanishing Christoffel symbols are

$$\Gamma^r_{rr} = \frac{1}{2}g^{rr}(\partial_r g_{rr}) = \alpha', \tag{2.32}$$

$$\Gamma^r_{\theta\theta} = \frac{1}{2}g^{rr}(-\partial_r g_{\theta\theta}) = -re^{-2\alpha}, \tag{2.33}$$

$$\Gamma^r_{\phi\phi} = \frac{1}{2}g^{rr}(-\partial_r g_{\phi\phi}) = -r\sin^2\theta e^{-2\alpha} = -re^{-2\alpha}, \tag{2.34}$$

$$\Gamma^\theta_{\theta r} = \frac{1}{2}g^{\theta\theta}(\partial_r g_{\theta\theta}) = \frac{1}{r}, \tag{2.35}$$

$$\Gamma^\phi_{\phi r} = \frac{1}{2}g^{\phi\phi}(\partial_r g_{\phi\phi}) = \frac{1}{2}\frac{1}{r^2\sin^2\theta}(2r\sin^2\theta) = \frac{1}{r}, \tag{2.36}$$

or are related to these by symmetry. There are also

$$\Gamma^\phi_{\phi\theta} = \frac{1}{2}g^{\phi\phi}(\partial_\theta g_{\phi\phi}) = \cot\theta, \tag{2.37}$$

$$\Gamma^\theta_{\phi\phi} = \frac{1}{2}g^{\theta\theta}(-\partial_\theta g_{\phi\phi}) = -\sin\theta\cos\theta, \tag{2.38}$$

whose partial derivative w.r.t θ does not vanish:

$$\partial_\theta\Gamma^\phi_{\phi\theta} = -\csc^2\theta = -1, \tag{2.39}$$

$$\partial_\theta\Gamma^\theta_{\phi\phi} = -\cos^2\theta + \sin^2\theta = 1. \tag{2.40}$$

Again, since the metric tensor is diagonal, to calculate the scalar curvature, the only relevant Ricci tensors are R_{rr}, $R_{\theta\theta}$, and $R_{\phi\phi}$.

$$\begin{aligned} R_{rr} &= \partial_i\Gamma^i_{rr} - \partial_r\Gamma^i_{ri} + \Gamma^i_{ij}\Gamma^j_{rr} - \Gamma^i_{rj}\Gamma^j_{ri} \\ &= \partial_r\Gamma^r_{rr} - \partial_r\Gamma^r_{rr} - \partial_r\Gamma^\theta_{r\theta} - \partial_r\Gamma^\phi_{r\phi} + (\Gamma^r_{rr})^2 + \Gamma^\theta_{\theta r}\Gamma^r_{rr} + \Gamma^\phi_{\phi r}\Gamma^r_{rr} - (\Gamma^r_{rr})^2 - (\Gamma^\theta_{r\theta})^2 - (\Gamma^\phi_{r\phi})^2 \\ &= \frac{2}{r^2} + 2\frac{\alpha'}{r} - \frac{2}{r^2} \\ &= 2\frac{\alpha'}{r}, \end{aligned} \tag{2.41}$$

$$\begin{aligned} R_{\theta\theta} &= \partial_i\Gamma^i_{\theta\theta} - \partial_\theta\Gamma^i_{\theta i} + \Gamma^i_{ij}\Gamma^j_{\theta\theta} - \Gamma^i_{\theta j}\Gamma^j_{\theta i} \\ &= \partial_r\Gamma^r_{\theta\theta} - \partial_\theta\Gamma^\phi_{\theta\phi} + \Gamma^r_{rr}\Gamma^r_{\theta\theta} + \Gamma^\theta_{\theta r}\Gamma^r_{\theta\theta} + \Gamma^\phi_{\phi r}\Gamma^r_{\theta\theta} - 2\Gamma^\theta_{\theta r}\Gamma^r_{\theta\theta} \\ &= -\partial_r(re^{-2\alpha}) + 1 - r\alpha' e^{-2\alpha} - 2e^{-2\alpha} + 2e^{-2\alpha} \\ &= -e^{-2\alpha} + r\alpha' e^{-2\alpha} + 1, \end{aligned} \tag{2.42}$$

and

$$\begin{aligned} R_{\phi\phi} &= \partial_i \Gamma^i_{\phi\phi} - \partial_\phi \Gamma^i_{\phi i} + \Gamma^i_{ij}\Gamma^j_{\phi\phi} - \Gamma^i_{\phi j}\Gamma^j_{\phi i} \\ &= \partial_r \Gamma^r_{\phi\phi} + \partial_\theta \Gamma^\theta_{\phi\phi} + \Gamma^r_{rr}\Gamma^r_{\phi\phi} + \Gamma^\theta_{\theta r}\Gamma^r_{\phi\phi} + \Gamma^\phi_{\phi r}\Gamma^r_{\phi\phi} - 2\Gamma^\phi_{\phi r}\Gamma^r_{\phi\phi} \\ &= -e^{-2\alpha} + r\alpha' e^{-2\alpha} + 1. \end{aligned} \tag{2.43}$$

Then,

$$\begin{aligned} R_{(3)} &= g^{rr}R_{rr} + g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi} \\ &= e^{-2\alpha}\left(2\frac{\alpha'}{r}\right) + \frac{2}{r^2}(-e^{-2\alpha} + r\alpha' e^{-2\alpha} + 1) \\ &= \frac{2}{r^2}\left[r\alpha' e^{-2\alpha} - e^{-2\alpha} + r\alpha' e^{-2\alpha} + 1\right] \\ &= \frac{2}{r^2}\left[1 - \frac{\mathrm{d}}{\mathrm{d}r}\left(re^{-2\alpha(r/R_0)}\right)\right]. \end{aligned} \tag{2.44}$$

3. Requiring the scalar curvature to be a constant, say $R_{(3)} = A$:

$$\begin{aligned} A = R_{(3)} &= \frac{2}{r^2}\left[1 - \frac{\mathrm{d}}{\mathrm{d}r}\left(re^{-2\alpha(r/R_0)}\right)\right] \\ \frac{\mathrm{d}}{\mathrm{d}r}\left(re^{-2\alpha(r/R_0)}\right) &= 1 - \frac{Ar^2}{2} \\ re^{-2\alpha(r/R_0)} + B &= r - \frac{Ar^3}{6} \\ e^{-2\alpha(r/R_o)} &= 1 - \frac{Ar^2}{6} - \frac{B}{r} \end{aligned} \tag{2.45}$$

Recognizing $\frac{A}{6} = \frac{k}{R_0^2}$ and $B = bR_0$ shall give us

$$e^{2\alpha(r/R_0)} = \frac{1}{1 - k\frac{r^2}{R_0^2} - b\left(\frac{r}{R_0}\right)^{-1}}. \tag{2.46}$$

As $b\left(\frac{r}{R_0}\right)$ is divergent at $r = 0$, the local flatness at this point requires $b = 0$ and hence,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1 - kr^2/R_0^2} + r^2\mathrm{d}\Omega^2\right]. \tag{2.47}$$

The $k = -1, 0, 1$ just corresponds to *hyperbolic*, *flat*, and *spherical* space, respectively. R_0 simply tells the spatial curvature.

4.

$$\frac{\partial \rho}{\partial t} = \dot{a}r = \frac{\dot{a}}{a}\rho, \tag{2.48}$$

$$\frac{\partial \rho}{\partial r} = a, \tag{2.49}$$

$$\frac{\partial T}{\partial t} = 1 + \frac{1}{2}\ddot{a}ar^2 + \frac{1}{2}\dot{a}^2r^2 = 1 + \frac{\ddot{a}}{2a}\rho^2 + \frac{\dot{a}^2}{2a^2}\rho^2, \tag{2.50}$$

$$\frac{\partial T}{\partial r} = \dot{a}ar = \dot{a}\rho. \tag{2.51}$$

The Jacobian is given by

$$J = \begin{pmatrix} \frac{\partial T}{\partial t} & \frac{\partial T}{\partial r} \\ \frac{\partial \rho}{\partial t} & \frac{\partial \rho}{\partial r} \end{pmatrix}. \tag{2.52}$$

Its determinant is given by

$$\begin{aligned} \det J &= \frac{\partial T}{\partial t}\frac{\partial \rho}{\partial r} - \frac{\partial T}{\partial r}\frac{\partial \rho}{\partial t} \\ &= \left(a + \frac{\ddot{a}}{2}\rho^2 + \frac{\dot{a}^2}{2a}\rho^2\right) - \frac{\dot{a}^2}{a}\rho^2 \\ &= \left(a + \frac{\ddot{a}}{2}\rho^2 - \frac{\dot{a}^2}{2a}\rho^2\right). \end{aligned} \tag{2.53}$$

Inverting the Jacobian give us

$$J^{-1} \equiv \begin{pmatrix} \frac{\partial t}{\partial T} & \frac{\partial t}{\partial \rho} \\ \frac{\partial r}{\partial T} & \frac{\partial r}{\partial \rho} \end{pmatrix} = \frac{1}{\det J}\begin{pmatrix} \frac{\partial \rho}{\partial r} & -\frac{\partial T}{\partial r} \\ -\frac{\partial \rho}{\partial t} & \frac{\partial T}{\partial t} \end{pmatrix}. \tag{2.54}$$

We can then read off the elements:

$$\frac{\partial t}{\partial T} = \frac{1}{\det J}\frac{\partial \rho}{\partial r} = \left(a + \frac{\ddot{a}}{2}\rho^2 - \frac{\dot{a}^2}{2a}\rho^2\right)^{-1} a \approx \left(1 - \frac{\ddot{a}}{2a}\rho^2 + \frac{\dot{a}^2}{2a^2}\rho^2\right), \tag{2.55}$$

$$\frac{\partial t}{\partial \rho} = -\frac{1}{\det J}\frac{\partial T}{\partial r} = -\left(a + \frac{\ddot{a}}{2}\rho^2 - \frac{\dot{a}^2}{2a}\rho^2\right)^{-1} \dot{a}\rho \approx -\frac{\dot{a}}{a}\rho\left(1 - \frac{\ddot{a}}{2a}\rho^2 + \frac{\dot{a}^2}{2a^2}\rho^2\right), \tag{2.56}$$

$$\frac{\partial r}{\partial T} = -\frac{1}{\det J}\frac{\partial \rho}{\partial t} = -\left(a + \frac{\ddot{a}}{2}\rho^2 - \frac{\dot{a}^2}{2a}\rho^2\right)^{-1} \frac{\dot{a}}{a}\rho \approx -\frac{\dot{a}}{a^2}\rho\left(1 - \frac{\ddot{a}}{2a}\rho^2 + \frac{\dot{a}^2}{2a^2}\rho^2\right), \tag{2.57}$$

$$\frac{\partial r}{\partial \rho} = \frac{1}{\det J}\frac{\partial T}{\partial t} = \left(a + \frac{\ddot{a}}{2}\rho^2 - \frac{\dot{a}^2}{2a}\rho^2\right)^{-1}\left(1 + \frac{\ddot{a}}{2a}\rho^2 + \frac{\dot{a}^2}{2a^2}\rho^2\right) \approx \frac{1}{a}\left(1 + \frac{\dot{a}^2}{a^2}\rho^2\right). \tag{2.58}$$

Thus,

$$\begin{aligned} \mathrm{d}t^2 &= \left(1 - \frac{\ddot{a}}{2a}\rho^2 + \frac{\dot{a}^2}{2a^2}\rho^2\right)^2 \mathrm{d}T^2 + \frac{\dot{a}^2}{a^2}\rho^2\left(1 - \frac{\ddot{a}}{2a}\rho^2 + \frac{\dot{a}^2}{2a^2}\rho^2\right)^2 \mathrm{d}\rho^2 \\ &\quad - 2\frac{\dot{a}}{a}\rho\left(1 - \frac{\ddot{a}}{2a}\rho^2 + \frac{\dot{a}^2}{2a^2}\rho^2\right)^2 \mathrm{d}\rho\mathrm{d}T \\ &\approx \left(1 - \frac{\ddot{a}}{a}\rho^2 + \frac{\dot{a}^2}{a^2}\rho^2\right)\mathrm{d}T^2 + \frac{\dot{a}^2}{a^2}\rho^2\mathrm{d}\rho^2 - 2\frac{\dot{a}}{a}\rho\mathrm{d}\rho\mathrm{d}T, \end{aligned} \tag{2.59}$$

and

$$\begin{aligned} \mathrm{d}r^2 &= \frac{\dot{a}^2}{a^4}\rho^2\left(1 - \frac{\ddot{a}}{2a}\rho^2 + \frac{\dot{a}^2}{2a^2}\rho^2\right)^2 \mathrm{d}T^2 + \frac{1}{a^2}\left(1 + \frac{\dot{a}^2}{a^2}\rho^2\right)^2 \mathrm{d}\rho^2 \\ &\quad - 2\frac{\dot{a}}{a^3}\rho\left(1 - \frac{\ddot{a}}{2a}\rho^2 + \frac{\dot{a}^2}{2a^2}\rho^2\right)\left(1 + \frac{\dot{a}^2}{a^2}\rho^2\right)\mathrm{d}\rho\mathrm{d}T \\ &\approx \frac{\dot{a}^2}{a^4}\rho^2\mathrm{d}T^2 + \frac{1}{a^2}\left(1 + 2\frac{\dot{a}^2}{a^2}\rho^2\right)\mathrm{d}\rho^2 - 2\frac{\dot{a}}{a^3}\rho\mathrm{d}\rho\mathrm{d}T \end{aligned} \tag{2.60}$$

and also,

$$\begin{aligned}
\frac{a^2 \mathrm{d}r^2}{1-kr^2/R_0^2} &= \frac{a^2}{1-k\rho^2/(a^2R_0^2)}\mathrm{d}r^2 \\
&\approx a^2\left(1+\frac{k\rho^2}{a^2R_0^2}\right)\mathrm{d}r^2 \\
&= \frac{\dot{a}^2}{a^2}\rho^2\mathrm{d}T^2 + \left(1+\frac{k\rho^2}{a^2R_0^2}+2\frac{\dot{a}^2}{a^2}\rho^2\right)\mathrm{d}\rho^2 - 2\frac{\dot{a}}{a}\rho\mathrm{d}\rho\mathrm{d}T,
\end{aligned} \tag{2.61}$$

where we have ignored terms in $\mathcal{O}(\rho^3)$ and higher. Thus,

$$\begin{aligned}
\mathrm{d}s^2 &= -\mathrm{d}t^2 + \frac{a^2\mathrm{d}r^2}{1-kr^2/R_0^2} + a^2r^2\mathrm{d}\Omega^2 \\
&\approx -\left(1-\frac{\dot{a}^2}{a^2}\rho^2\right)\mathrm{d}T^2 + \left(1+\frac{k\rho^2}{a^2R_0^2}+\frac{\dot{a}^2}{a^2}\rho^2\right)\mathrm{d}\rho^2 + \rho^2\mathrm{d}\Omega^2.
\end{aligned} \tag{2.62}$$

To extract the Newtonian effective potential, we can match with the Newtonian weak field metrics $g_{00} = -(1+2\Phi)$ component, and read off

$$1+2\Phi = 1-\frac{\dot{a}^2}{a^2}\rho^2, \tag{2.63}$$

or

$$\Phi(\rho) = -\frac{1}{2}\frac{\dot{a}^2}{a^2}\rho^2. \tag{2.64}$$

Notice ρ really measures the physical distance and the effective potential captures the tidal effect from the cosmic acceleration by a local observer.

2.2 Geodesics from a Lagrangian

1.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\alpha}\right) &= \frac{\partial\mathcal{L}}{\partial x^\alpha} \\
-2\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(g_{\mu\alpha}\dot{x}^\mu\right) &= g_{\mu\nu,\alpha}\dot{x}^\mu\dot{x}^\nu \\
g_{\mu\alpha}\ddot{x}^\mu &= -\frac{1}{2}g_{\mu\nu,\alpha}\dot{x}^\mu\dot{x}^\nu + g_{\mu\alpha,\nu}\dot{x}^\mu\dot{x}^\nu \\
g_{\mu\alpha}\ddot{x}^\mu &= -\frac{1}{2}\left(g_{\mu\nu,\alpha}-g_{\mu\alpha,\nu}-g_{\nu\alpha,\mu}\right)\dot{x}^\mu\dot{x}^\nu \\
\ddot{x}^\beta &= -\frac{1}{2}g^{\beta\alpha}\left(g_{\mu\nu,\alpha}-g_{\mu\alpha,\nu}-g_{\nu\alpha,\mu}\right)\dot{x}^\mu\dot{x}^\nu \\
\ddot{x}^\beta &= -\Gamma^\beta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu,
\end{aligned} \tag{2.65}$$

where we used the trick to replace $g_{\mu\alpha,\nu}$ with $\frac{1}{2}\left(g_{\mu\alpha,\nu}+g_{\nu\alpha,\mu}\right)$ on the fourth line because it is contracted with $\dot{x}^\mu\dot{x}^\nu$ which is symmetric in μ and ν. This is exactly the geodesic equation.

2. Note that the Lagrangian by definition is a function of some coordinate and its first derivative.

$$\begin{aligned}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}\lambda} &= \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\lambda} - \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\dot{x}^\mu\right)\\ &= \frac{\partial\mathcal{L}}{\partial\lambda} + \frac{\partial\mathcal{L}}{\partial x^\mu}\dot{x}^\mu + \frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\ddot{x}^\mu - \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\right)\dot{x}^\mu - \frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\ddot{x}^\mu\\ &= \frac{\partial\mathcal{L}}{\partial x^\mu}\dot{x}^\mu - \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\right)\dot{x}^\mu\\ &= 0,\end{aligned} \tag{2.66}$$

where we invoked the Euler-Lagrange equation in the last line. Thus, the Hamiltonian $\mathcal{H}$ is a constant along the geodesics.

2.3 Christoffel symbols from a Lagrangian

The metric tensor is given by

$$g_{\mu\nu} = diag(-1, a^2(t)\delta_{ij}), \tag{2.67}$$

and thus,

$$\mathcal{L} \equiv -g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \dot{t}^2 - a^2\delta_{ij}\dot{x}^i\dot{x}^j. \tag{2.68}$$

Applying the Euler-Lagrange equation for time coordinate shall give us

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial\mathcal{L}}{\partial\dot{t}}\right) &= \frac{\partial\mathcal{L}}{\partial t}\\ 2\frac{\mathrm{d}\dot{t}}{\mathrm{d}\lambda} &= -2a\dot{a}\delta_{ij}\dot{x}^i\dot{x}^j\\ \frac{\mathrm{d}^2t}{\mathrm{d}\lambda^2} &= -a\dot{a}\delta_{ij}\dot{x}^i\dot{x}^j.\end{aligned} \tag{2.69}$$

Comparing this with the geodesic equation of 0-th component:

$$\frac{\mathrm{d}^2t}{\mathrm{d}\lambda^2} = -\Gamma^0_{\mu\nu}\dot{x}^\mu\dot{x}^\nu, \tag{2.70}$$

one can read off

$$\Gamma^0_{ij} = a\dot{a}\delta_{ij} \tag{2.71}$$

and

$$\Gamma^0_{00} = \Gamma^0_{0i} = \Gamma^0_{i0} = 0. \tag{2.72}$$

A bit of caution of notation abuse here: $\dot{a} \equiv \frac{\mathrm{d}a}{\mathrm{d}t}$ while $\dot{x}^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}$.

Similarly, we can apply the Euler-Lagrange equation for k-th coordinate,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^k}\right) &= \frac{\partial\mathcal{L}}{\partial x^k}\\ -2\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(a^2\delta_{ik}\dot{x}^i\right) &= 0\\ a^2\delta_{ik}\ddot{x}^i &= -2a\dot{a}\dot{t}\delta_{ik}\dot{x}^i\\ \ddot{x}^k &= -2\frac{\dot{a}}{a}\dot{t}\delta^k_i\dot{x}^i\end{aligned} \tag{2.73}$$

Comparing this with the geodesic equation of k-th component:

$$\frac{\mathrm{d}^2 x^k}{\mathrm{d}\lambda^2} = -\Gamma^k_{\mu\nu}\dot{x}^\mu \dot{x}^\nu, \tag{2.74}$$

one can read off

$$\Gamma^k_{0i} + \Gamma^k_{i0} = 2\frac{\dot{a}}{a}\delta^k_i \tag{2.75}$$

or

$$\Gamma^k_{0i} = \Gamma^k_{i0} = \frac{\dot{a}}{a}\delta^k_i. \tag{2.76}$$

Also,

$$\Gamma^k_{00} = \Gamma^k_{ij} = 0. \tag{2.77}$$

This result should not come as much of a surprise since we used the Euler-Lagrange equation to derive the geodesic equation in the first place. Of course, the two approaches should give the same result.

2.4 Geodesics in de Sitter space

1. The Lagrangian is

$$\mathcal{L} = \left(1 - \frac{r^2}{R^2}\right)\dot{t}^2 - \left(1 - \frac{r^2}{R^2}\right)^{-1}\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2 \equiv \left(1 - \frac{r^2}{R^2}\right)\dot{t}^2 - \left(1 - \frac{r^2}{R^2}\right)^{-1}\dot{r}^2 - r^2\dot{\Omega}^2, \tag{2.78}$$

where we used $\mathrm{d}\Omega^2 \equiv \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2$ such that $\dot{\Omega}^2 = \dot{\theta}^2 + \sin^2\theta\dot{\phi}^2$. Now, since the Lagrangian has no apparent dependencies on t and Ω, the two coordinates must have associated conserved quantity.

For t coordinate, we call this conserved quantity total energy E. Using the Euler-Lagrange equation [1],

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{t}}\right) &= 0 \\ \sqrt{E} &= \left(1 - \frac{r^2}{R^2}\right)\dot{t} \\ E &\equiv \left(1 - \frac{r^2}{R^2}\right)^2 \dot{t}^2 \end{aligned} \tag{2.79}$$

For the solid angle Ω coordinate, we call this conserved quantity total angular momentum L:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{\Omega}}\right) &= 0 \\ L &\equiv r^2\dot{\Omega} \\ L^2 = r^4\dot{\Omega}^2 &= r^4(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2). \end{aligned} \tag{2.80}$$

[1] The reason of using $\sqrt{E}$ instead of E to be the LHS is because we want the Lagrangian and the energy to have same dimensions (though they are both dimensionless in this question due to the Lagrangian in the book is written in a way that the mass of the particle has been set to unity.)

2. Rewriting the Lagrangian with the conserved quantities E and L:

$$\mathcal{L} = \left(1 - \frac{r^2}{R^2}\right)^{-1} (E - \dot{r}^2) - \frac{L^2}{r^2}. \tag{2.81}$$

The problem 2.3 also told us that for a massive particle,

$$g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -1 \tag{2.82}$$

and thus,

$$\begin{aligned} \mathcal{L} &= 1 \\ \left(1 - \frac{r^2}{R^2}\right)^{-1} (E - \dot{r}^2) - \frac{L^2}{r^2} &= 1 \\ \dot{r}^2 = E - \left(1 + \frac{L^2}{r^2}\right)\left(1 - \frac{r^2}{R^2}\right) &\equiv E - V_{\text{eff}}(r). \end{aligned} \tag{2.83}$$

Thus, the radial motion is governed by the effective potential

$$V_{\text{eff}}(r) = \left(1 + \frac{L^2}{r^2}\right)\left(1 - \frac{r^2}{R^2}\right) = 1 - \frac{L^2}{R^2} + \frac{L^2}{r^2} - \frac{r^2}{R^2}. \tag{2.84}$$

The sketch of this potential is shown in Fig. 2.1.

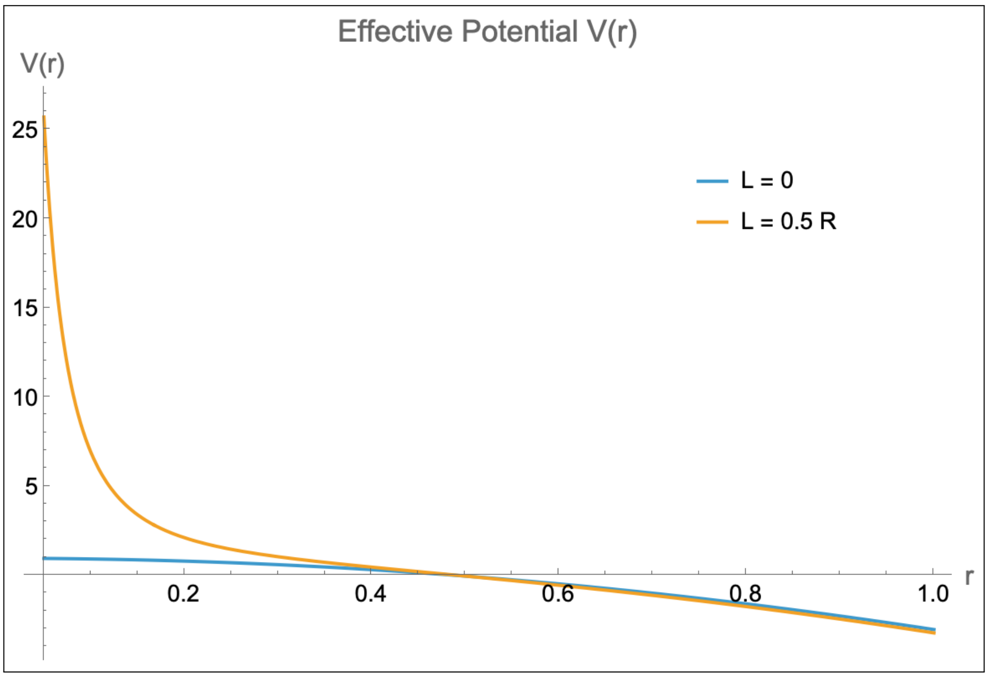


Fig. 2.1: The effective potential for $L = 0$ and $L = 0.5R$

3. From Eq. (2.83),

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = \sqrt{E - V_{\text{eff}}(r)} = \sqrt{E - 1 + \frac{r^2}{R^2}}, \tag{2.85}$$

where we set $L = 0$ since the particle has only radial velocity. Thus,

$$\begin{aligned} \int_0^r \frac{\mathrm{d}r'}{\sqrt{E - 1 + \frac{r'^2}{R^2}}} &= \tau \\ R\sinh^{-1}\left(\frac{r}{R\sqrt{E-1}}\right) &= \tau \\ r &= R\sqrt{E-1}\sinh\left(\frac{\tau}{R}\right). \end{aligned} \tag{2.86}$$

The $\Delta\tau$ is apparently finite when $\Delta r = R$.

To find the trajectory $r(t)$, we notice that

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \sqrt{E}\left(1 - \frac{r^2}{R^2}\right)^{-1}. \tag{2.87}$$

Thus,

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}r}{\mathrm{d}\tau} = \frac{1}{\sqrt{E}}\left(1 - \frac{r^2}{R^2}\right)\sqrt{E - 1 + \frac{r^2}{R^2}}. \tag{2.88}$$

Then,

$$\begin{aligned} \Delta t &= \int_0^R \frac{\sqrt{E}\mathrm{d}r}{\left(1 - \frac{r^2}{R^2}\right)\sqrt{E - 1 + \frac{r^2}{R^2}}} \\ &= \left[R\tanh^{-1}\left(\frac{\sqrt{E}r}{R\sqrt{E - 1 + \frac{r^2}{R^2}}}\right)\right]_0^R \\ &= \infty. \end{aligned} \tag{2.89}$$

2.5 Distances

1. The proper distance is given by $\ell_0 \equiv S_{k=0}(\chi) \equiv \chi$ for a flat universe ($k = 0$). From Eq. (2.67) of the textbook:

$$\begin{aligned}
\ell_0 \equiv \chi(z) &= \int_{t_1}^{t_0} \frac{\mathrm{d}t}{a(t)} \\
&= \int_0^z \frac{\mathrm{d}z}{H(z)} \\
&= \frac{1}{H_0\sqrt{\Omega_m}} \int_0^z \mathrm{d}z a^{3/2} \qquad \text{(cf. Eq. (2.144) of textbook)} \\
&= \frac{1}{H_0\sqrt{\Omega_m}} \int_0^z \mathrm{d}z'(1+z')^{-3/2} \text{(cf. Eq. (2.63) of textbook)} \\
&= \frac{2}{H_0\sqrt{\Omega_m}} \left(1 - \frac{1}{\sqrt{1+z}}\right) \\
&= 3t_0 \left(1 - \frac{1}{\sqrt{1+z}}\right) \qquad \text{(cf. Eq. (2.151) of textbook)},
\end{aligned} \tag{2.90}$$

where we also plugged in $\Omega_m = 1$ for a matter-dominated universe in the last line. Note that this equation can also be inverted to get an expression for $(1+z)$:

$$1 + z = \left(1 - \frac{\ell_0}{3t_0}\right)^{-2}. \tag{2.91}$$

From Eq. (2.71) of the textbook, we then have

$$d_L(z) = (1+z)d_M(z) \equiv (1+z)\ell_0 = \ell_0 \left(1 - \frac{\ell_0}{3t_0}\right)^{-2}. \tag{2.92}$$

When $\ell_0 \to 3t_0$, $d_L \to \infty$. This corresponds $z \to \infty$ or $a(t) \to 0$, which is exactly the Big Bang singularity. This implies everything was infinitesimally close at the singularity.

2. By Eq. (2.83) of the textbook,

$$d_A(z) = \frac{d_M(z)}{1+z} \equiv \frac{\ell_0}{1+z} = \frac{2}{H_0}\left(1 - \frac{1}{\sqrt{1+z}}\right)\frac{1}{1+z}. \tag{2.93}$$

A plot of the angular size as a function of the redshift z is shown in Fig. 2.2. Indeed, the angular size of these objects at first decreases with distance, but then becomes larger beyond a critical distance.

To find the maximum of the angular diameter distance (i.e., the minimum of the angular size) of the object, we take the derivative with respect to the redshift z and set the derivative to 0:

$$0 = \frac{\mathrm{d}d_A}{\mathrm{d}z} = \frac{2}{H_0}\left[-\frac{1}{(1+z_{\min})^2} + \frac{3}{2}\frac{1}{(1+z_{\min})^{5/2}}\right]. \tag{2.94}$$

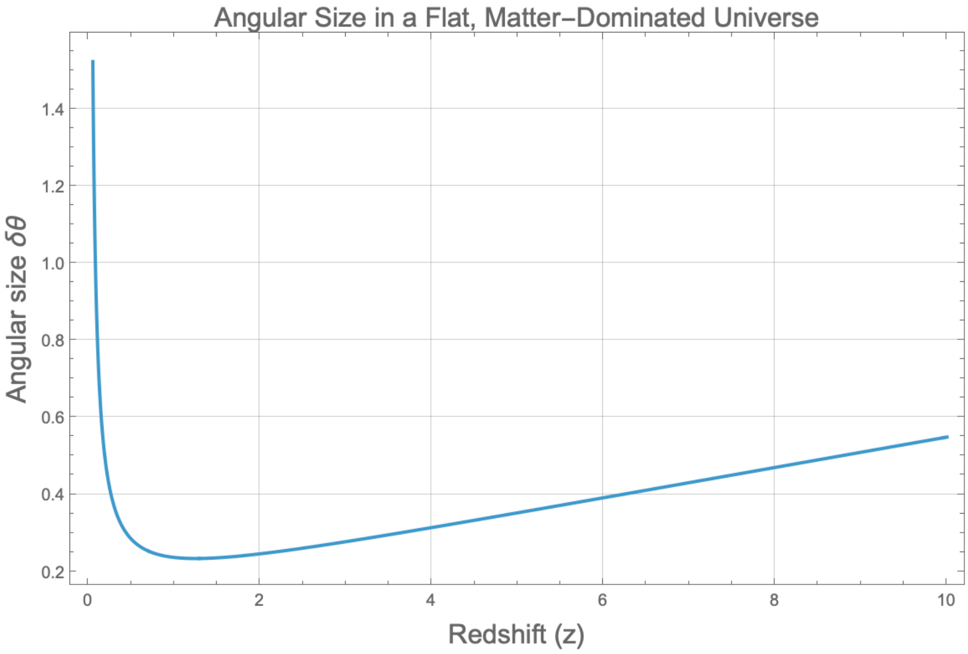


Fig. 2.2: The angular size $\delta\theta$ as a function of redshift z.

Thus,

$$z_{\min} = \frac{5}{4}. \tag{2.95}$$

At this redshift, the angular diameter distance is

$$d_A(z_{\min}) = \frac{8}{27H_0}. \tag{2.96}$$

From Eq. (2.93), the angular diameter distance vanishes at two points: $z \to 0$ and $z \to \infty$, or in other words, the angular size $\delta\theta = \frac{D}{d_A}$ for an object with physical size D diverges. The $z \to 0$ singularity corresponds to objects that are right on our eyes at present time, which of course, occupies an infinite angular size. The $z \to \infty$ corresponds to the Big Bang singularity, at which point everything was so close to each other, and thus, also takes an infinite angular size, as it spreads out the whole universe at present time.

2.6 Flatland cosmology

1. The metric tensor is $g_{\mu\nu} = (-1, a^2\delta_{ij})$. The calculation and reasoning are no different from the 3-spatial dimensions case. Thus, the only nonzero components are the same as Eq. (2.45) of the textbook

$$\begin{aligned} \Gamma^0_{ij} &= a\dot{a}\delta_{ij}, \\ \Gamma^i_{0j} &= \frac{\dot{a}}{a}\delta^i_j, \end{aligned} \tag{2.97}$$

except that now even Γ^i_{jk} vanishes because $g_{ij} \equiv a^2\delta_{ij}$ has no spatial dependence.

2. Since the Christoffel symbols are the same as the $3+1$ dimension case, so are the Ricci tensor except that $\delta^i_i = 2$ instead of 3 now. From Eq. (2.121) of the textbook, we have

$$\begin{aligned} R_{00} &= -\partial_0 \Gamma^i_{0i} - \Gamma^i_{0j}\Gamma^j_{0i} \\ &= -\partial_0 \left(\frac{\dot{a}}{a}\right)\delta^i_i - \delta^i_j \delta^j_i \left(\frac{\dot{a}}{a}\right)^2 \\ &= -2\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 - 2\left(\frac{\dot{a}}{a}\right)^2 \\ &= -2\frac{\ddot{a}}{a}. \end{aligned} \tag{2.98}$$

Similarly, from Eq. (2.122) of the textbook, we have

$$\begin{aligned} R_{ij} &= \partial_0 \Gamma^0_{ij} + \Gamma^l_{l0}\Gamma^0_{ij} - \Gamma^0_{il}\Gamma^l_{j0} - \Gamma^l_{i0}\Gamma^0_{jl} \\ &= \left(\dot{a}^2 + a\ddot{a} + 2\dot{a}^2 - \dot{a}^2 - \dot{a}^2\right)\delta_{ij} \\ &= \left(\dot{a}^2 + a\ddot{a}\right)\delta_{ij} \end{aligned} \tag{2.99}$$

3. First note that the Ricci scalar is given by

$$R = g^{\mu\nu}R_{\mu\nu} = -R_{00} + \frac{1}{a^2}\delta^{ij}R_{ij} = 2\frac{\ddot{a}}{a} + \frac{2}{a^2}\left(\dot{a}^2 + a\ddot{a}\right) = 2\left[2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right]. \tag{2.100}$$

From the Einstein equations,

$$\begin{aligned} G_{00} \equiv R_{00} - \frac{1}{2}Rg_{00} &= 8\pi G T_{00} \\ -2\frac{\ddot{a}}{a} + \left[2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right] &= 8\pi G\rho \\ \boxed{\left(\frac{\dot{a}}{a}\right)^2 = 8\pi G\rho}&, \end{aligned} \tag{2.101}$$

and

$$\begin{aligned} G_{ij} \equiv R_{ij} - \frac{1}{2}Rg_{ij} &= 8\pi G T_{ij} \\ \left(\dot{a}^2 + a\ddot{a}\right)\delta_{ij} - \left[2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right]a^2\delta_{ij} &= 8\pi G a^2 P\delta_{ij} \\ \boxed{\frac{\ddot{a}}{a} = -8\pi G P}&. \end{aligned} \tag{2.102}$$

4. Note that the conservation of the energy density is given by $\nu = 0$

$$\begin{aligned} \nabla_\mu T^\mu_0 = 0 \implies \partial_\mu T^\mu_0 + \Gamma^\mu_{\mu\lambda}T^\lambda_0 - \Gamma^\lambda_{\mu 0}T^\mu_\lambda &= 0 \\ \partial_0 T^0_0 + \Gamma^i_{i0}T^0_0 - \Gamma^i_{j0}T^j_i &= 0 \\ \dot{\rho} + 2\frac{\dot{a}}{a}(\rho + P) &= 0. \end{aligned} \tag{2.103}$$

Let $P = w\rho$. Then,

$$\frac{\dot{\rho}}{\rho} = -2\frac{\dot{a}}{a}(1+w). \tag{2.104}$$

Thus, we have

$$\boxed{\rho \propto a^{-2(1+w)}}\ , \tag{2.105}$$

from which we can read off $n = 2(1+w)$.

For a pressureless fluid, $w = 0$, and thus, $\rho \propto a^{-2}$. The energy density decreases as the inverse of area growth, as expected.

5. Using the $2+1$ dimension version of the first Friedmann equation (cf. Eq. (2.101)), and the scaling of the energy density (cf. Eq. (2.105)), we have

$$\left(\frac{\dot{a}}{a}\right)^2 \propto a^{-2(1+w)}$$
$$a^{1+w} \propto t$$
$$\boxed{a \propto t^{1/(1+w)}}\ . \tag{2.106}$$

We can read off $q = \frac{1}{1+w}$.

2.7 Friedmann universe

1. The Friedmann equation is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho}{3} - \frac{k}{a^2 R_0^2}, \tag{2.107}$$

where $\rho = \frac{\rho_0}{a^{(1+3w)}} + \Lambda$ (cf. Eq.(2.107) of the textbook, with Λ explicitly split out). Hence, we can rewrite the Friedmann equation as

$$\frac{1}{2}\dot{a}^2 - \frac{\rho_0}{6}\frac{1}{a^{(1+3w)}} + \frac{K}{2} - \frac{\Lambda}{6}a^2 = 0. \tag{2.108}$$

The first the term can be recognized as the kinetic term, while the rest is the potential

$$\boxed{V(a) = -\frac{\rho_0}{6}\frac{1}{a^{(1+3w)}} + \frac{K}{2} - \frac{\Lambda}{6}a^2}\ . \tag{2.109}$$

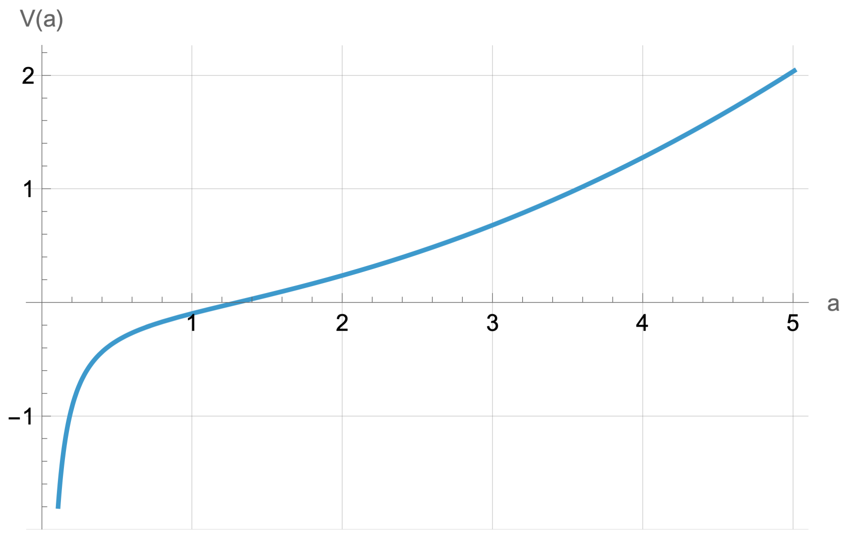
The sketch of $V(a)$ for the 3 cases are shown in Fig. 2.3.

Note at $a_{\max}$, $\dot{a} = 0$ and hence, by Eq. (2.108), $a_{\max}$ is determined by where $V(a_{\max}) = 0$.

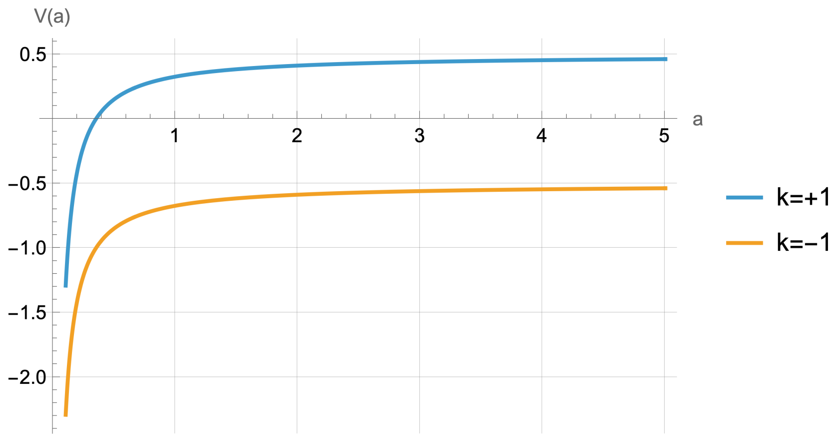
(i) $\boldsymbol{k = 0, \Lambda < 0}$**:**

$$-\frac{\rho_0}{6}\frac{1}{a_{\max}^{(1+3w)}} - \frac{\Lambda}{6}a_{\max}^2 = 0,$$
$$\boxed{a_{\max} = \left(-\frac{\rho_0}{\Lambda}\right)^{\frac{1}{3(1+w)}}}\ . \tag{2.110}$$

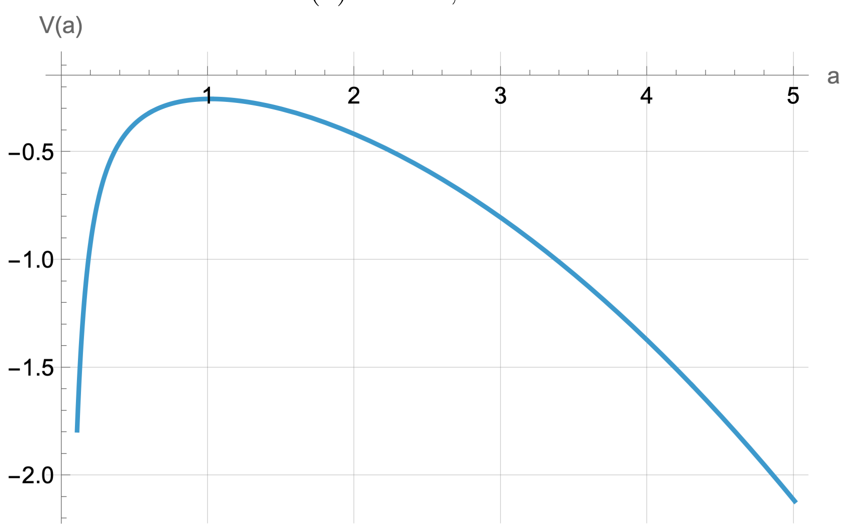
Since $\Lambda < 0$, the RHS has real solution, and hence, a physical $a_{\max}$.



(i) $k = 0$, $\Lambda < 0$



(ii) $k = \pm 1$, $\Lambda = 0$



(iii) $k = 0$, $\Lambda > 0$

Fig. 2.3: Potential $V(a)$ for the three cases.

(ii) $\boldsymbol{k = \pm 1,\ \Lambda = 0}$:

$$-\frac{\rho_0}{6}\frac{1}{a_{\text{max}}^{(1+3w)}} \pm \frac{1}{2R_0^2} = 0,$$
$$\boxed{a_{\text{max}} = \left(\pm\frac{\rho_0 R_0^2}{3}\right)^{\frac{1}{1+3w}}}, \tag{2.111}$$

for $k = \pm 1$. However, only for $k = +1$, does this have a real solution and hence, a physical a_{max}.

(iii) $\boldsymbol{k = 0,\ \Lambda > 0}$:

The formula is already given by Eq. (2.110). However now, since $\Lambda < 0$, there is no real solution and hence, no physical a_{max}.

These results can be confirmed with the sketch as well. The a_{max} only exists for those $V(a)$ that can cross zero point.

2. From Eq. (2.157) of the textbook,

$$a'' + \frac{1}{R_0^2}a = \frac{1}{6}(\rho - 3P)\,a^3,$$
$$a'' + \frac{1}{R_0^2}a = \frac{\rho}{6}(1 - 3w)\,a^3,$$
$$\boxed{a'' + \frac{1}{R_0^2}a = \frac{\rho_0}{6}(1 - 3w)\,a^{-3w}}, \tag{2.112}$$

where $\rho = \frac{\rho_0}{a^{(1+3w)}}$ is again invoked.

The solution can be simply verified by explicitly plugging into the differential equation.

Since the trig function sin can be at most 1,

$$A = a_{\text{max}} = \left(\frac{\rho_0 R_0^2}{3}\right)^{\frac{1}{1+3w}}. \tag{2.113}$$

$$0 \equiv a(\eta = 0) \implies 0 = \sin(B) \implies B = \pi n, \quad n = 0,\ \pm 1,\ \pm 2,\ \cdots. \tag{2.114}$$

We can choose the principal $n = 0$ and set $\boxed{B = 0}$. Then,

$$a(\eta) = \left(\frac{\rho_0 R_0^2}{3}\right)^{\frac{1}{1+3w}} \left[\sin\left(\frac{1+3w}{2}\frac{\eta}{R_0}\right)\right]^{\frac{2}{1+3w}}. \tag{2.115}$$

From Eq. (2.115), the big crunch happens when

$$\sin\left(\frac{1+3w}{2}\frac{\eta}{R_0}\right) = 0. \tag{2.116}$$

(i) **Pressureless matter ($w = 0$):**
This happens when

$$\frac{\eta}{2R_0} = \pi n \implies \eta = 2\pi n R_0, \quad n = 0,\ \pm 1,\ \pm 2,\ \cdots. \tag{2.117}$$

As the $n = 0$ principal can be interpreted as the Big Bang, the $n = 1$ principal then should be interpreted as the Big Crunch. Hence,

$$\boxed{\eta = 2\pi R_0}. \tag{2.118}$$

By the definition of the conformal time η, since the photon travels on null path, the conformal time tells exactly the distance the photon can travel. In the case of a pressureless matter universe, $\eta = 2\pi R_0$. Hence, the photon travels exactly one circle before the universe ends.

(ii) **Radiation ($w = \frac{1}{3}$):**
This happens when

$$\frac{\eta}{R_0} = \pi n \implies \eta = \pi n R_0, \quad n = 0,\ \pm 1,\ \pm 2,\ \cdots. \tag{2.119}$$

As the $n = 0$ principal can be interpreted as the Big Bang, the $n = 1$ principal then should be interpreted as the Big Crunch. Hence,

$$\boxed{\eta = \pi R_0}. \tag{2.120}$$

In the case of a radiation universe, $\eta = \pi R_0$. Hence, the photon travels exactly half circle before the universe ends.

2.8 Einstein's biggest blunder

1. For a static solution to exist, the scale factor a must remain a constant in the whole history of universe, which requires all of its time derivative to vanish. The 2nd Friedmann equation says

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P). \tag{2.121}$$

Since both density $\rho > 0$ and pressure $P > 0$, the RHS remains negative. In the LHS, as the scale factor $a > 0$ always, it must always be that $\ddot{a} < 0$, and it never vanishes. Therefore, there is no static solution to the Einstein equations.

2. With the addition of a cosmological constant Λ, the Einstein equations are modified to

$$G^{\mu}{}_{\nu} = 8\pi G T^{\mu}{}_{\nu} - \Lambda g^{\mu}{}_{\nu}. \tag{2.122}$$

Hence, for a pressureless matter only universe, the 1st Friedmann equation becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_m - \frac{k}{a^2 R_0^2} - \frac{\Lambda}{3}. \tag{2.123}$$

The 2nd Friedmann equation (with $P = 0$) becomes

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho_m + \frac{\Lambda}{3}. \tag{2.124}$$

For the $\ddot{a} = 0$, we then have

$$\boxed{\Lambda = 4\pi G \rho_m}\ . \tag{2.125}$$

Plugging this back into the fist Friedmann equation Eq. (2.123), and require it to vanish as well, we then have

$$\frac{4\pi G}{3}\rho_m = \frac{k}{a^2 R_0^2}. \tag{2.126}$$

Since $\rho_m > 0$, $a > 0$, and $R_0 > 0$, it must be that $k > 0$, and hence, the universe is positively curved ($k = 1$), and its spatial curvature is

$$\boxed{R_0 = \sqrt{\frac{3}{4\pi G a^2 \rho_m}}}\ . \tag{2.127}$$

This universe is static provided that ρ_m is a constant such that any higher time derivatives of a vanish identically.

3. From the continuity equation, we know that (cf. Eq. (2.108) of the textbook), $\rho_m \propto a^{-3}$. Combined this with the hints, we have

$$\begin{gathered} 1 + \delta(t) \propto a^{-3}, \\ \boxed{\delta(t) \propto a^{-3} - 1 = (1 + \epsilon(t))^{-3} - 1 \approx -3\epsilon(t)}\ . \end{gathered} \tag{2.128}$$

Hence, the two perturbations are related to each other.

Now, plugging this into the 2nd Friedmann equation Eq. (2.124) with $\Lambda = 4\pi G \rho_{m,0}$, we have

$$\begin{aligned} \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}\rho_{m,0}\delta(t) = -\frac{4\pi G}{3}\rho_{m,0}(a^{-3} - 1) \\ \ddot{\epsilon} &= \frac{\Lambda}{3}(a - a^{-2}) \approx \frac{\Lambda}{3}(1 + \epsilon - 1 + 2\epsilon) = \Lambda\epsilon, \end{aligned} \tag{2.129}$$

which has solution

$$\boxed{\epsilon = Ae^{\sqrt{\Lambda}t} + Be^{-\sqrt{\Lambda}t}}\ . \tag{2.130}$$

As the $\delta(t) \propto -3\epsilon(t)$ (cf. Eq. (2.128)), both perturbations grow exponentially with time, and this static universe is unstable.

2.9 The accelerating universe

1. Note that the deceleration parameter is just $q(t) \equiv -\frac{\ddot{a}a}{\dot{a}^2} = -\frac{\text{2nd FE}}{\text{1st FE}}$. The 1st FE with no radiation can be written as (cf. Eq. (2.204) of the textbook)

$$\left(\frac{\dot{a}}{a}\right)^2 \equiv H^2 = H_0^2\left[\Omega_m a^{-3} + \Omega_\Lambda + (1 - \Omega_0)a^{-2}\right] \tag{2.131}$$

and the 2nd FE with pressureless matter and a positive cosmological constant Λ can be written as (cf. Eq. (2.137) of the textbook and Eq. (2.124))

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho_m + \frac{\Lambda}{3} = H_0^2\left(-\frac{1}{2}\Omega_m a^{-3} + \Omega_\Lambda\right). \tag{2.132}$$

Hence,

$$q(t) \equiv -\frac{\ddot{a}a}{\dot{a}^2} = -\frac{\text{2nd FE}}{\text{1st FE}} = \boxed{\frac{\frac{1}{2}\Omega_m a^{-3} - \Omega_\Lambda}{\Omega_m a^{-3} + \Omega_\Lambda + (1-\Omega_0)a^{-2}}}. \tag{2.133}$$

The plot is displayed in Fig. 2.4.

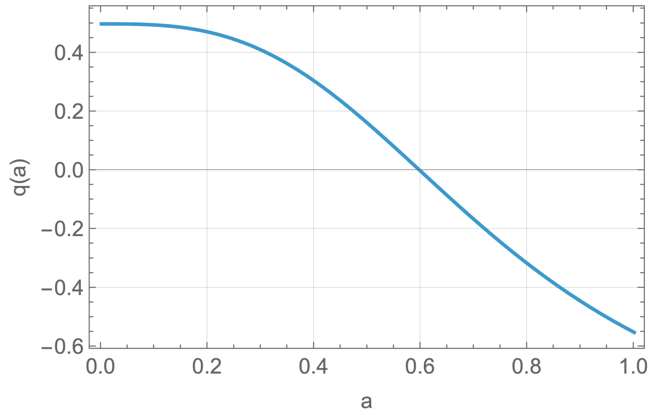


Fig. 2.4: $q(t)$ as a function of $a(t)$ for our universe.

Plugging in $a(t_0) = 1$ today into Eq. (2.133), we see

$$\boxed{q_0 = \frac{1}{2}\Omega_m - \Omega_\Lambda}. \tag{2.134}$$

For our universe, we get

$$\boxed{q_0 = -0.55}, \tag{2.135}$$

which is a negative number so our universe is accelerating its expansion.

2. Note that if we take the time derivative of the deceleration parameter directly and divided by the Hubble parameter:

$$\frac{\dot{q}}{H} = -\frac{\dddot{a}a^2}{\dot{a}^3} - \frac{\ddot{a}a}{\dot{a}^2} + 2\frac{\ddot{a}^2a^2}{\dot{a}^4} = -J + q + 2q^2. \tag{2.136}$$

To get an easy form for taking time derivative of q, we first take the time derivative of the Hubble parameter

$$\dot{H} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{a}}{a}\right) = \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = \frac{\ddot{a}}{a} - H^2. \tag{2.137}$$

Hence, the 2nd FE can be written as

$$\frac{\ddot{a}}{a} = \dot{H} + H^2. \tag{2.138}$$

Then, since we know $q(t) \equiv -\frac{\ddot{a}a}{\dot{a}^2} = -\frac{\text{2nd FE}}{\text{1st FE}}$, we can write it as

$$q = -\frac{\dot{H} + H^2}{H^2} = -\frac{\dot{H}}{H^2} - 1 \tag{2.139}$$

This allows us the

$$\dot{q} = -\frac{\ddot{H}}{H^2} + 2\frac{\dot{H}^2}{H^3} \tag{2.140}$$

Hence,

$$J = -\frac{\dot{q}}{H} + q + 2q^2 = \frac{\ddot{H}}{H^3} - 2\frac{\dot{H}^2}{H^4} - \frac{\dot{H}}{H^2} - 1 + 2\left(-\frac{\dot{H}}{H^2} - 1\right)^2 = 1 + 3\frac{\dot{H}}{H^2} + \frac{\ddot{H}}{H^3}. \tag{2.141}$$

Taking the time derivative of the 1st FE in the form of Eq. (2.136) of the textbook successively and also invoke the continuity equation ($\dot{\rho} = -3H\rho$) for matter:

$$2H\dot{H} = \frac{8\pi G}{3}\dot{\rho} + 2\frac{k}{a^2R_0^2}H = -8\pi GH\rho + 2\frac{k}{a^2R_0^2}H, \tag{2.142}$$

$$\dot{H} = -4\pi G\rho + \frac{k}{a^2R_0^2}. \tag{2.143}$$

Taking Another time derivative and invoking the continuity equation again:

$$\ddot{H} = -4\pi G\dot{\rho} + 2\frac{k}{a^2R_0^2}H = 12\pi GH\rho - 2\frac{k}{a^2R_0^2}H. \tag{2.144}$$

Plugging these back into Eq. (2.141), we have

$$\boxed{J = 1 + \frac{1}{a^2H^2}\frac{k}{R_0^2}}\,. \tag{2.145}$$

3. At the point of matter-cosmological constant equality $\rho_{m,m\Lambda} = \rho_\Lambda$

$$\frac{\rho_\Lambda}{\rho_{m,0}} = \frac{\rho_{m,m\Lambda}}{\rho_{m,0}} = a_{m\Lambda}^{-3} = (1 + z_{m\Lambda})^3$$
$$\frac{\Omega_\Lambda}{1 - \Omega_\Lambda} = (1 + z_{m\Lambda})^3$$
$$\boxed{z_{m\Lambda} = \left(\frac{\Omega_\Lambda}{1 - \Omega_\Lambda}\right)^{1/3} - 1}\,. \tag{2.146}$$

This is the point that the universe starts to be dominated by the cosmological constant. In our universe, with $\Omega_\Lambda = 0.7$, we have

$$\boxed{z_{m\Lambda} \approx 0.33}\,. \tag{2.147}$$

4. The point universe start accelerating is defined by

$$\dot{q}_{acc} = 0,\ \ddot{q}_{acc} < 0. \tag{2.148}$$

Since $a > 0$ always, this is the point

$$\ddot{a}_{acc} = 0. \tag{2.149}$$

Plugging into the 2nd FE, one has

$$\begin{gathered}
-\frac{4\pi G}{3}\rho_{m,acc} + \frac{\Lambda}{3} = 0 \\
\rho_{m,0}a^{-3} - 2\rho_\Lambda = 0 \\
\boxed{z_{acc} = \left(2\frac{\rho_\Lambda}{\rho_{m,0}}\right)^{1/3} - 1 = \left(2\frac{\Omega_\Lambda}{\Omega_m}\right)^{1/3} - 1 = \left(\frac{2-2\Omega_m}{\Omega_m}\right)^{1/3} - 1}.
\end{gathered} \tag{2.150}$$

For our universe, plugging $\Omega_m = 0.3$,

$$\boxed{z_{acc} \approx 0.67} \tag{2.151}$$

is the redshift at which the universe begins accelerating.

5. From 1st FE,

$$\begin{aligned}
H^2 &= H_0^2\left[\Omega_m a^{-3} + (1-\Omega_m)\right] \\
\dot{a} &= H_0\sqrt{\Omega_m a^{-1} + (1-\Omega_m)a^2}.
\end{aligned} \tag{2.152}$$

Define $u^2 = a^3$, then

$$\mathrm{d}a = \frac{2}{3}u^{-1/3}\mathrm{d}u. \tag{2.153}$$

Integrating Eq. (2.152),

$$\begin{aligned}
\frac{3}{2}H_0\sqrt{1-\Omega_m}t &= \frac{3}{2}\int_0^a \frac{\mathrm{d}a'}{\sqrt{\left(\frac{\Omega_m}{1-\Omega_m}\right)a'^{-1} + a'^2}} \\
&= \int_0^u \frac{\mathrm{d}u'}{u^{1/3}\sqrt{\left(\frac{\Omega_m}{1-\Omega_m}\right)u'^{-2/3} + u'^{4/3}}} \\
&= \int_0^u \frac{\mathrm{d}u'}{\sqrt{\left(\frac{\Omega_m}{1-\Omega_m}\right) + u'^2}} \\
&= \left[\sinh^{-1}\left(\frac{u'}{\sqrt{\frac{\Omega_m}{1-\Omega_m}}}\right)\right]_0^u \\
&= \sinh^{-1}\left(\sqrt{\frac{1-\Omega_m}{\Omega_m}}u\right) \\
&= \sinh^{-1}\left(\sqrt{\frac{1-\Omega_m}{\Omega_m}}a^{3/2}\right)
\end{aligned} \tag{2.154}$$

Hence,

$$\boxed{a(t) = \left(\frac{\Omega_m}{1-\Omega_m}\right)^{1/3} \sinh^{2/3}\left(\frac{3}{2}H_0\sqrt{1-\Omega_m}t\right)}\,, \tag{2.155}$$

from which we can read off

$$\boxed{A = \left(\frac{\Omega_m}{1-\Omega_m}\right)^{1/3}}\,, \tag{2.156}$$

and

$$\boxed{\alpha = \frac{3}{2}H_0\sqrt{1-\Omega_m}}\,. \tag{2.157}$$

- At early universe, we can Taylor expand the $\sinh(x) \to x$. Hence,

$$a \propto t^{2/3}, \tag{2.158}$$

which is the correct limiting behavior for a matter-only Einstein-de Sitter universe (cf. Eq.(2.147) of the textbook).

At late universe, we can use the exponential form of $\sinh(x) = \frac{e^x - e^{-x}}{2}$ and note that the e^x dominates the late time behavior:

$$a \propto e^{H_0\sqrt{\Omega_\Lambda}t}, \tag{2.159}$$

which is the correct limiting behavior for a cosmological constant-only de Sitter universe (cf. Eq.(2.147) of the textbook).

-

$$\dot{a} = \frac{2}{3}A\alpha(\sinh(\alpha t))^{-1/3}\cosh(\alpha t). \tag{2.160}$$

$$\ddot{a} = \frac{2}{9}A\alpha^2(\sinh(\alpha t))^{-4/3}(\cosh(2\alpha t) - 2). \tag{2.161}$$

$$\dddot{a} = \frac{8}{27}A\alpha^3(\sinh(\alpha t))^{-7/3}\cosh^3(\alpha t). \tag{2.162}$$

Then, we have

$$q \equiv -\frac{\ddot{a}a}{\dot{a}^2} = -\frac{1}{2}\frac{\cosh(2\alpha t) - 2}{\cosh^2(\alpha t)} = \boxed{\frac{3}{2}\operatorname{sech}^2(\alpha t) - 1}\,, \tag{2.163}$$

where A is given Eq. (2.156) and α is given by Eq. (2.157). For the jerk,

$$J \equiv \frac{\dddot{a}a^2}{\dot{a}^3} = \boxed{1}\,, \tag{2.164}$$

as expected for a flat universe.

- Invert the solution and plugging $a(t_0) = 1$. We can estimate the age of our universe as

$$t_0 = \frac{2}{3}\frac{1}{H_0\sqrt{1-\Omega_m}}\sinh^{-1}\left(\sqrt{\frac{1-\Omega_m}{\Omega_m}}\right) \approx \boxed{14.0 \text{ Gyr}}\,, \tag{2.165}$$

where we have plugged in $\Omega_m = 0.3$ and $H_0 = h \times (9.777 \text{ Gyr})^{-1}$ with $h = 0.674$ (cf. Appendix C.2.3 of the textbook).

6. We first need to figure out the comoving distance χ. From Eq. (2.67) of the textbook,

$$\chi = \int_{t_1}^{t_0} \frac{\mathrm{d}t}{a(t)} = \int_a^1 \frac{\mathrm{d}a'}{Ha'^2} = \frac{1}{H_0}\int_a^1 \frac{\mathrm{d}a'}{a'^2\sqrt{\Omega_m a'^{-3} + (1-\Omega_m)}}. \tag{2.166}$$

Let $s = \frac{1}{a} = 1+z$. Then, $-a^2\mathrm{d}s = \mathrm{d}a$. We have

$$\chi = \frac{1}{H_0}\int_1^s \frac{\mathrm{d}s'}{\sqrt{\Omega_m s'^3 + (1-\Omega_m)}}, \tag{2.167}$$

which unfortunately has no closed form in general and has to be integrated numerically. Using Eq. (2.71) of the textbook and the fact that in flat universe $d_M(z) = \chi(z)$, we have

$$d_L(z) = (1+z)\chi(z), \tag{2.168}$$

where χ is given by Eq. (2.167).

Doing the integration numerically, the figure is displayed in Fig. 2.5.

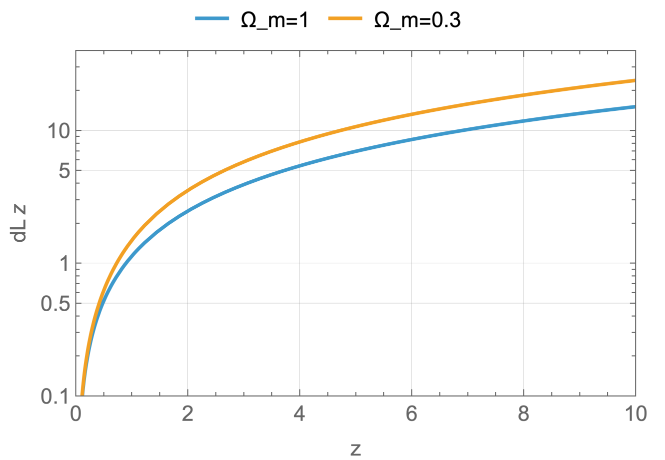


Fig. 2.5: Luminosity distance (in units of c/H_0) as a function of redshift in a flat universe.

At $z = 0.5$, the luminosity distances (in units of c/H_0) of the two models are about

$$\boxed{d_L(z) = \begin{cases} 0.55 & \Omega_m = 1 \\ 0.66 & \Omega_m = 0.3 \end{cases}}. \tag{2.169}$$

Hence, the accuracy needs to be at least

$$\boxed{\frac{0.66 - 0.55}{0.66} \approx 16.7\ \%} \tag{2.170}$$

to distinguish the two models.

2.10 Phantom dark energy

1. • Invoking continuity equation, we have

$$\rho_X \propto a^{-3(1+w_X)}. \tag{2.171}$$

Since $w_X < -1$ for phantom dark energy, the power is positive. Thus, ρ_X increases with the scale factor a.

Next, we need to prove that the universe is, once expanding, always expanding such that the scale factor increases with time. From the 1st FE,

$$\frac{H^2}{H_0^2} = \Omega_m a^{-3} + \Omega_X a^{-3(1+w_X)}. \tag{2.172}$$

For an expanding universe to turn to shrinking, there must be at turn point $H = 0$. However, as $-3(1 + w_X) > 0$, the RHS never goes to 0.

Therefore, as the energy density of the phantom dark energy ρ_X increases with the scale factor a, and a increases with time, we conclude that the energy density of the phantom dark energy must always increases with time.

•

$$\begin{aligned}
\Omega_X(a) &\equiv \frac{\rho_X(a)}{\rho_{\text{crit}}(a)} \\
&= \frac{\rho_X(a)}{\rho_{X,0}}\frac{\rho_{X,0}}{\rho_{\text{crit},0}}\frac{\rho_{\text{crit},0}}{\rho_{\text{crit}}(a)} \\
&= a^{-3(1+w_X)}\Omega_{X,0}\frac{H_0^2}{H^2} \\
&= \Omega_{X,0}\left(\Omega_{X,0} + \Omega_{m,0}a^{3w_X}\right)^{-1} \\
&= \boxed{\left(1 + \frac{\Omega_{m,0}}{\Omega_{X,0}}a^{3w_X}\right)^{-1}}\,.
\end{aligned} \tag{2.173}$$

• Inverting the above,

$$a = \left[\frac{\Omega_{X,0}}{\Omega_{m,0}}(\Omega_X^{-1} - 1)\right]^{\frac{1}{3w_X}}. \tag{2.174}$$

Plugging the values $\Omega_{X,0} = 0.75$, $\Omega_{m,0} = 1 - \Omega_{X,0} = 0.25$, $w_X = -2$, and $\Omega_X = 0.999$, we have

$$\boxed{a \approx 2.63}\,. \tag{2.175}$$

2. Integrating the 1st FE, we have

$$\begin{aligned}
\Delta t &= \frac{1}{H_*}\int_{a_*}^{\infty} \frac{\mathrm{d}a}{\sqrt{\Omega_{m,*}a^{-1} + \Omega_{X,*}a^{-1-3w_X}}} \\
&\approx \frac{1}{H_*}\int_{a_*}^{\infty} \frac{\mathrm{d}a}{\sqrt{\Omega_{X,*}a^{-1-3w_X}}} \\
&= \frac{1}{H_*\sqrt{\Omega_{X,*}}}\int_{a_*}^{\infty} \mathrm{d}a a^{(1+3w_X)/2} \\
&= -\frac{2}{(1+3w_X)H_*\sqrt{\Omega_{X,*}}} a_*^{3(1+w_X)/2},
\end{aligned} \tag{2.176}$$

where we have used the fact that as a^{-1-3w_X} diverges as $a \to \infty$, the contribution from $a^{-1-3w_X} \gg a^{-1}$ for large a.

We can solve for a_* using Eq. (2.174) and the fact that at t_*, we have $\Omega_{X,*} = \Omega_{m,*} = \frac{1}{2}$.

$$a_* = \left(\frac{\Omega_{X,0}}{\Omega_{m,0}}\right)^{\frac{1}{3w_X}}. \tag{2.177}$$

Hence,

$$\boxed{\Delta t = -\frac{2\sqrt{2}}{(1+3w_X)H_*}\left(\frac{\Omega_{X,0}}{\Omega_{m,0}}\right)^{\frac{1+w_X}{2w_X}}}. \tag{2.178}$$

The RHS is a perfectly positive finite physical quantity. Hence, in such a universe, "Big Rip" is real.

3. Recall that the redshift of the wavelength is given by (cf. Eq.(2.56) of the textbook)

$$\lambda_{\text{rip}} = \frac{a(t_{\text{rip}})}{a(t_{\text{CMB}})}\lambda_{\text{CMB}}. \tag{2.179}$$

As $a(t_{\text{rip}}) \to \infty$ and both $a(t_{\text{CMB}})$ and λ_{CMB} are finite, we conclude that $\boxed{\lambda_{\text{rip}} \to \infty}$. Hence, the wavelength of CMB photons would be infinitely long, and the universe becomes really "dark".

Chapter 3

The Hot Big Bang

Exercise 3.1

In the relativistic limit ($x \to 0$), we have

$$J_\pm(0) = \int_0^\infty d\xi \frac{\xi^3}{e^\xi \pm 1} \tag{3.1}$$

The quickest way to compute $J_-(0)$ is to invoke the identity from Eq. (D.34) in the textbook appendix (which also provides the fastest route to $I_-(0)$):

$$\zeta(s) = \frac{1}{\Gamma(s)} \int dx \frac{x^{s-1}}{e^x - 1}. \tag{3.2}$$

Thus,

$$J_-(0) = \Gamma(4)\zeta(4) = 3!\zeta(4) = \boxed{6\zeta(4)}\,. \tag{3.3}$$

Similarly,

$$\begin{aligned} J_+(0) &= \int_0^\infty d\xi \frac{\xi^3}{e^\xi + 1} \\ &= \int_0^\infty d\xi \frac{\xi^3}{e^\xi - 1} - 2\int_0^\infty d\xi \frac{\xi^3}{e^{2\xi} - 1} \\ &= J_-(0) - 2 \times \left(\frac{1}{2}\right)^4 \int_0^\infty d(2\xi)\, \frac{(2\xi)^3}{e^{2\xi} - 1} \\ &= J_-(0) - 2 \times \left(\frac{1}{2}\right)^4 J_-(0) \\ &= \boxed{\frac{7}{8} J_-(0)}\,. \end{aligned} \tag{3.4}$$

Exercise 3.2

We begin with a quick and rough derivation. Starting from Eq. (3.11), Eq. (3.12), and Eq. (3.27) of the textbook, and noting that in the non-relativistic limit ($x \gg 1$), we find:

We shall derive these in a quick and dirty way. Starting from Eq. (3.11), Eq. (3.12), and Eq. (3.27) of the textbook, and noting that in the non-relativistic limit ($x \gg 1$),

$$\frac{\mathrm{d}n}{\mathrm{d}T} = \frac{g}{2\pi^2} \int_0^\infty \mathrm{d}pp^2 \frac{\sqrt{p^2+m^2}}{T^2} e^{-\sqrt{p^2+m^2}/T} = \frac{\rho}{T^2}. \tag{3.5}$$

Thus, we conclude $\rho = T^2 \frac{\mathrm{d}n}{\mathrm{d}T}$. On the other hand,differentiating the number density expression as in Eq. (3.31) of the textbook with respect to T yields

$$\rho = T^2 \frac{\mathrm{d}n}{\mathrm{d}T} = T^2 \left(\frac{m}{T^2} n + \frac{3}{2} \frac{n}{T} \right) = \boxed{mn + \frac{3}{2} nT}\,. \tag{3.6}$$

Comparing this with the non-relativistic expansion $E(p) = \sqrt{m^2+p^2} \approx m + \frac{p^2}{2m}$ and referring to Eq. (3.9) of the textbook, it's evident that the mn term comes from the rest mass contribution, while the $\frac{3}{2}nT$ term comes from the kinetic energy $\frac{p^2}{2m}$.

Now, in the non-relativistic limit,

$$P(T) \approx \frac{1}{3m} \frac{g}{(2\pi)^3} \int \mathrm{d}^3 p f(p,T) p^2. \tag{3.7}$$

Comparing with the result above, we obtain

$$P(T) = \frac{1}{3m} \times \frac{3}{2} nT \times 2m = \boxed{nT}\,. \tag{3.8}$$

Exercise 3.3

$$T\mathrm{d}S = \mathrm{d}U + P\mathrm{d}V - \mu \mathrm{d}N. \tag{3.9}$$

Using $S \equiv sV$, $U \equiv \rho V$, and $N \equiv nV$, this becomes

$$\begin{aligned} T\,\mathrm{d}(sV) &= \mathrm{d}(\rho V) + P\mathrm{d}V - \mu\,\mathrm{d}(nV)\,, \\ (Ts - \rho - P - \mu n)\mathrm{d}V + V\left(T\frac{\mathrm{d}s}{\mathrm{d}t} - \frac{\mathrm{d}\rho}{\mathrm{d}t} + \mu \frac{\mathrm{d}n}{\mathrm{d}t} \right) \mathrm{d}t &= 0. \end{aligned} \tag{3.10}$$

Each bracketed term must vanish independently. The first gives

$$\boxed{s = \frac{\rho + P - \mu n}{T}}\,. \tag{3.11}$$

Now, recall the continuity equation:

$$\begin{aligned} \dot{\rho} + 3H(\rho + P) &= 0 \\ \dot{\rho} &= -3H(Ts + \mu n), \end{aligned} \tag{3.12}$$

where we used the result above.

The vanishing of the second bracket implies

$$\begin{aligned}
\dot{s} &= \frac{1}{T}\dot{\rho} - \frac{\mu}{T}\dot{n}, \\
\dot{s} &= -3H\left(s + \frac{\mu}{T}n\right) - \frac{\mu}{T}\dot{n}, \\
a^3\dot{s} + 3a^2\dot{a}s &= -3a^2\dot{a}\frac{\mu}{T}n - a^3\frac{\mu}{T}\dot{n}, \\
&\boxed{\frac{\mathrm{d}(sa^3)}{\mathrm{d}t} = -\frac{\mu}{T}\frac{\mathrm{d}(na^3)}{\mathrm{d}t}}\,.
\end{aligned} \tag{3.13}$$

The total entropy is conserved if the right-hand side of Eq. (3.13) vanishes. This occurs either when $\mu/T \to 0$ (i.e., the chemical potential is negligible), or when $\frac{\mathrm{d}(na^3)}{\mathrm{d}t} = 0$ (i.e., the total particle number is conserved).

Exercise 3.4

Eq. (3.82)–(3.84) from the textbook are still valid, except we now treat $X_e = 1$. Then,

$$\begin{aligned}
(T_{\rm dec}T_0)^{3/2} &= \frac{\pi^2}{2\zeta(3)}\frac{H_0\sqrt{\Omega_m}}{\eta\sigma_T}, \\
(T_0^2(1+z_{\rm dec}))^{3/2} &= \frac{\pi^2}{2\zeta(3)}\frac{H_0\sqrt{\Omega_m}}{\eta\sigma_T}, \\
&\boxed{z_{\rm dec} = \left(\frac{\pi^2}{2\zeta(3)}\frac{H_0\sqrt{\Omega_m}}{\eta\sigma_T}\right)^{2/3} T_0^{-2} - 1 \approx 34.6}\,,
\end{aligned} \tag{3.14}$$

where we have plugged in the values $H_0 = 0.674 \times 2.133 \times 10^{-33}$ eV, $\Omega_m = 0.315$, $\eta \approx 6 \times 10^{-10}$, $\sigma_T = 2 \times 10^{-3}$ MeV^{-2}, and $T_0 \approx 0.235$ meV (cf. Appendix C.2.3 of the textbook).

3.1 Chemical potential of electrons

1. Using Eq. (3.4) and Eq. (3.8) of the textbook for fermions, we obtain

$$\begin{aligned}
n - \bar{n} &= \frac{g}{2\pi^2}\int_0^\infty \mathrm{d}p \left[\frac{p^2}{e^{\left(\sqrt{p^2+m^2}-\mu\right)/T}+1} - \frac{p^2}{e^{\left(\sqrt{p^2+m^2}+\mu\right)/T}+1}\right] \\
&= \frac{g}{2\pi^2}T^3\int_0^\infty \mathrm{d}\xi \left[\frac{\xi^2}{e^{\xi-y}+1} - \frac{\xi^2}{e^{\xi+y}+1}\right] \\
&\equiv \frac{g}{2\pi^2}T^3\mathcal{I}(\xi),
\end{aligned} \tag{3.15}$$

where we used that the chemical potential of the antiparticle is opposite in sign to that of the particle. We also introduced the dimensionless variables $\xi \equiv p/T$ and $y \equiv \mu/T$, and took the relativistic limit $m \ll T$.

We now apply a linear change of variable: $\xi' = \xi - y$ in the first term and $\xi' = \xi + y$ in the second. The integral becomes

$$\begin{aligned}
\mathcal{I}(\xi') &= \int_{-y}^{\infty} \mathrm{d}\xi' \frac{(\xi'+y)^2}{e^{\xi'}+1} - \int_{y}^{\infty} \mathrm{d}\xi' \frac{(\xi'-y)^2}{e^{\xi'}+1} \\
&= \int_{0}^{\infty} \mathrm{d}\xi' \frac{(\xi'+y)^2}{e^{\xi'}+1} + \int_{-y}^{0} \mathrm{d}\xi' \frac{(\xi'+y)^2}{e^{\xi'}+1} - \int_{0}^{\infty} \mathrm{d}\xi' \frac{(\xi'-y)^2}{e^{\xi'}+1} + \int_{0}^{y} \mathrm{d}\xi' \frac{(\xi'-y)^2}{e^{\xi'}+1} \\
&= \int_{0}^{\infty} \mathrm{d}\xi' \left[\frac{(\xi'+y)^2 - (\xi'-y)^2}{e^{\xi'}+1}\right] + \int_{0}^{y} \mathrm{d}\xi' \left[\frac{(-\xi'+y)^2}{e^{-\xi'}+1} + \frac{(\xi'-y)^2}{e^{\xi'}+1}\right] \\
&= 4y \int_{0}^{\infty} \mathrm{d}\xi' \frac{\xi'}{e^{\xi'}+1} + \int_{0}^{y} \mathrm{d}\xi' (\xi'-y)^2 \\
&= \frac{\pi^2}{3} y + \frac{y^3}{3},
\end{aligned} \tag{3.16}$$

where I flipped the sign of ξ' in the second term of the second line. Substituting this result into Eq. (3.15) gives

$$\boxed{n - \bar{n} = \frac{gT^3}{6\pi^2}\left[\pi^2\left(\frac{\mu}{T}\right) + \left(\frac{\mu}{T}\right)^3\right]}\,. \tag{3.17}$$

2. From subsection 3.1.3 of the textbook, we know that the baryon-to-photon ratio $\eta \equiv \frac{n_b}{n_\gamma} \equiv \frac{n_b - n_{\bar{b}}}{n_\gamma}$ remains conserved after the epoch of electron-positron annihilation. Moreover, since the universe today is electrically neutral, we have

$$\eta \equiv \frac{n_B}{n_\gamma} \approx \frac{n_p - n_{\bar{p}}}{n_\gamma} = \frac{n_e - n_{\bar{e}}}{n_\gamma}.$$

Then, using Eq. (3.76) of the textbook, we get

$$n_e - n_{\bar{e}} \approx n_b = \eta n_\gamma = \eta \times \frac{2\zeta(3)}{\pi^2} T^3. \tag{3.18}$$

Comparing this result with Eq. (3.17), and using $g_e = 2$, we find

$$\frac{1}{6\zeta(3)}\left[\pi^2\left(\frac{\mu_e}{T}\right) + \left(\frac{\mu_e}{T}\right)^3\right] = \eta. \tag{3.19}$$

Plugging in the observed values $\eta \approx 6 \times 10^{-10}$ yields

$$\left(\frac{\mu_e}{T}\right)\left[\pi^2 + \left(\frac{\mu_e}{T}\right)^2\right] \approx 4 \times 10^{-9}. \tag{3.20}$$

Since the right-hand side is significantly less than 1, and the bracketed term on the left-hand side is definitely greater than 1, it follows that $\frac{\mu_e}{T} \ll 1$. Thus, we can neglect the $\left(\frac{\mu_e}{T}\right)^2$ term and conclude that $\boxed{\frac{\mu_e}{T} \approx 10^{-10} \sim 10^{-9}}$.

3.2 Conservation of entropy

1. From Eq. (3.4) of the textbook with $\mu = 0$, we have

$$\frac{\partial f}{\partial T} = \frac{e^{E/T}}{(e^{E/T} \pm 1)^2} \frac{E}{T^2}, \tag{3.21}$$

and

$$\frac{\partial f}{\partial p} = -\frac{e^{E/T}}{(e^{E/T} \pm 1)^2} \frac{1}{T} \frac{\mathrm{d}E}{\mathrm{d}p} = -\frac{e^{E/T}}{(e^{E/T} \pm 1)^2} \frac{p}{ET}. \tag{3.22}$$

Inspecting Eq. (3.21) and Eq. (3.22), we observe that

$$\boxed{\frac{\partial f}{\partial T} = -\frac{E^2}{Tp} \frac{\partial f}{\partial p}}. \tag{3.23}$$

From Eq. (3.10) of the textbook,

$$\begin{aligned}
\frac{\partial P}{\partial T} &= \frac{g}{(2\pi)^3} \int \mathrm{d}^3 p \left(\frac{\partial f}{\partial T}\right) \frac{p^2}{3E} \\
&= -\frac{g}{(2\pi)^3} \int d\Omega \int \mathrm{d}p \left(\frac{\partial f}{\partial p}\right) \frac{p^3 E}{3T} \\
&= \frac{g}{(2\pi)^3} \int d\Omega \left[\int \mathrm{d}p \left(\frac{p^2 E}{T} + \frac{p^4}{3ET}\right) f - \left[f \frac{p^3 E}{3T}\right]_0^\infty\right] \qquad \text{(IBP)} \\
&= \frac{g}{(2\pi)^3} \left[\int \mathrm{d}^3 p \left(\frac{E}{T} + \frac{p^2}{3ET}\right) f - 0\right] \qquad (f \to 0 \text{ as } p \to \infty) \\
&= \boxed{\frac{\rho + P}{T}}. \qquad \text{(cf. Eq. (3.9) and Eq. (3.10) of textbook)}
\end{aligned} \tag{3.24}$$

2. Note that for a massive spin-1 boson, its internal degrees of freedom is $g_X = 3$. For the neutrino-X system, we have

$$g_{*S} = \begin{cases} 3 + \frac{7}{8} \times 2 = \frac{19}{4} & T \gtrsim m_X, \\ \frac{7}{8} \times 2 = \frac{7}{4} & T < m_X. \end{cases} \tag{3.25}$$

Similarly, since entropy is separately conserved for the neutrino-X system,

$$g_{*S}(aT_\nu)^3 = const, \tag{3.26}$$

throughout the annihilation process. Thus, for the neutrino, aT_ν now should increases by a factor of $(\frac{19}{7})^{\frac{1}{3}}$, while the photon thermal bath has the energy transferred from the electron-positron annihilation and aT_γ thus, increases by a factor of $(\frac{11}{4})^{\frac{1}{3}}$ as the textbook discussed. Thus, the present neutrino temperature in this case is

$$\left(\frac{7}{19}\right)^{\frac{1}{3}} T_{\nu,0} = \left(\frac{4}{11}\right)^{\frac{1}{3}} T_{\gamma,0}$$

$$\boxed{T_{\nu,0} = \left(\frac{76}{77}\right)^{\frac{1}{3}} T_{\gamma,0}}. \tag{3.27}$$

3.3 Degenerate neutrinos

1. Assuming $\mu_\nu > 0$, from Eq. (3.9) of the textbook, we have

$$\rho(T) = \frac{g_\nu}{(2\pi)^3}\int \mathrm{d}^3 p f p = \frac{g_\nu}{2\pi^2}\int \mathrm{d}p \frac{p^3}{e^{\frac{p-\mu_\nu}{T}}+1}. \tag{3.28}$$

Now, note that the Heaviside step function $H(x)$ can be defined through

$$H(x) = \lim_{k\to\infty}\frac{1}{1+e^{-2kx}}. \tag{3.29}$$

This suggests

$$\begin{aligned}
\rho_\nu(T) &= \frac{g_\nu}{2\pi^2}\int \mathrm{d}p \frac{p^3}{1+e^{-2\left(\frac{\mu_\nu}{T}\right)\left(\frac{1}{2}-\frac{p}{2\mu_\nu}\right)}} \\
&\approx \frac{g_\nu}{2\pi^2}\int_0^\infty \mathrm{d}p p^3 H\left(\frac{1}{2}-\frac{p}{2\mu_\nu}\right) \qquad \left(\mu_\nu \gg T \implies \frac{\mu_\nu}{T}\to\infty\right) \\
&= \frac{g_\nu}{2\pi^2}\int_0^{\mu_\nu} \mathrm{d}p p^3 \\
&= \frac{g_\nu \mu_\nu^4}{8\pi^2}.
\end{aligned} \tag{3.30}$$

For antineutrinos, we have $\mu_{\bar\nu} = -\mu_\nu < 0$, by assumption. The effect is to change the argument of the Heaviside step function in the above derivation to $H\left(-\frac{1}{2}-\frac{p}{2\mu_\nu}\right)$ or in other words, in the integral vanishes unless $p < -\mu_\nu < 0$. However, since momentum is always positive, the integral always vanishes. Thus,

$$\rho_{\bar\nu} \approx 0. \tag{3.31}$$

Note that the above conclusion would be reversed if the neutrinos have $\mu_\nu < 0$. Thus, the combined energy density of degenerate neutrinos and antineutrinos can be expressed as

$$\boxed{\rho_\nu + \rho_{\bar\nu} \approx \frac{g_\nu |\mu_\nu|^4}{8\pi^2}}. \tag{3.32}$$

2. Eq. (3.62) of the textbook still holds:

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma. \tag{3.33}$$

The total energy density of degenerate neutrinos and antineutrinos is

$$\rho_\nu + \rho_{\bar\nu} \approx N_{\text{eff}}\frac{g_\nu |\mu_\nu|^4}{8\pi^2}. \tag{3.34}$$

Then,

$$\begin{aligned}
\rho_\nu + \rho_{\bar{\nu}} &\leq \rho_{\text{crit}} \\
N_{\text{eff}} \frac{g_\nu |\mu_\nu|^4}{8\pi^2} &\lesssim 6 \times 10^3 T_\gamma^4 = 6 \times 10^3 \times \left(\frac{11}{4}\right)^{4/3} T_\nu^4 \\
\boxed{\frac{|\mu_\nu|}{T_\nu} \lesssim \left[8\pi^2 \times 6 \times 10^3 \times \left(\frac{11}{4}\right)^{4/3} \frac{1}{g_\nu N_{\text{eff}}}\right]^{1/4} \approx 23.5}\,, \qquad (3.35)
\end{aligned}$$

where in the last step, we have plugged in the values $g_\nu = 2$ and $N_{\text{eff}} = 3$.

3. The degenerate neutrinos contribute extra component of energy density to the universe. From Friedmann equation (cf. Eq. (3.129) of the textbook), this larger energy density implies a larger Hubble expansion rate H as the universe was still radiation dominated at this point. This then implies an earlier neutron decoupling and then freeze-out, i.e. discussion following Eq.(3.120) of the textbook. This says there are more neutrons to protons, and then there are more deuterium formed, and hence, an enhanced helium-4 abundance (This is actually already discussed in the textbook in the paragraph at the end of the subsection *Helium.*).

3.4 Massive neutrinos

1. Simply from Eq. (3.12) of the textbook, and recall that neutrino is a fermion with internal degree of freedom $g_\nu = 2$:

$$\begin{aligned}
\rho_\nu &= \frac{1}{\pi^2} \int_0^\infty \mathrm{d}p \frac{p^2 \sqrt{p^2 + m_\nu^2}}{\exp\left[\sqrt{p^2 + m_\nu^2}/T_\nu\right] + 1} \\
&\approx \frac{1}{\pi^2} \int_0^\infty \mathrm{d}p \frac{p^2 \sqrt{p^2 + m_\nu^2}}{e^{\frac{p}{T_\nu}} + 1} \qquad (3.36) \\
&= \boxed{\frac{T_\nu^4}{\pi^2} \int_0^\infty \mathrm{d}\xi \frac{\xi^2 \sqrt{\xi^2 + m_\nu^2/T_\nu^2}}{e^\xi + 1}}\,.
\end{aligned}$$

2.

$$\begin{aligned}
\rho_\nu &= \frac{T_\nu^4}{\pi^2}\int_0^\infty \mathrm{d}\xi \frac{\xi^3\sqrt{1+m_\nu^2/(\xi^2 T_\nu^2)}}{e^\xi+1} \\
&\approx \frac{T_\nu^4}{\pi^2}\int_0^\infty \mathrm{d}\xi \frac{\xi^3+\xi\frac{m_\nu^2}{2T_\nu^2}}{e^\xi+1} \\
&= \rho_{\nu 0} + \frac{m_\nu^2}{2T_\nu^2}\frac{T_\nu^4}{\pi^2}\int_0^\infty \mathrm{d}\xi\xi\left(\frac{1}{e^\xi-1}-\frac{2}{e^{2\xi}-1}\right) \\
&= \rho_{\nu 0} + \frac{m_\nu^2}{2T_\nu^2}\frac{T_\nu^4}{\pi^2}\xi(2)\Gamma(2)\left[1-2\times\left(\frac{1}{2}\right)^2\right] \\
&= \rho_{\nu 0} + \frac{T_\nu^4}{24}\frac{m_\nu^2}{T_\nu^2} \\
&= \boxed{\rho_{\nu 0}\left(1+\frac{5}{7\pi^2}\frac{m_\nu^2}{T_\nu^2}\right)},
\end{aligned} \tag{3.37}$$

where I have used $\rho_{\nu 0} = \frac{7\pi^2}{120}T_\nu^4$ in the last line.

3. Following the discussion of the exact solution of two-component universe in Subsection 2.3.6 of the textbook, the radiation contribute no source term to the second Friedmann equation. Therefore, the massive neutrinos can leave imprints on the CMB anisotropies only if it behaves as a non-relativistic matter at the point of photon decoupling. We also know that the present $T_{\nu,0} = 1.95$ K. Note that after the neutrino decoupling ($T \sim 1$ MeV), which happens much before the photon decoupling ($T \sim 0.25$ eV), the neutrino $g_{*S}(T_\nu)$ does not change. Invoking the consequence of entropy conservation Eq. (3.54) of the textbook:
$$g_{*S}(T_\nu)T_\nu^3 a^3 = \text{const}, \tag{3.38}$$
we have $T \propto a^{-1} \propto (1+z)$. Thus, we can deduce that at the point of photon decoupling:
$$T_{\nu,\text{dec}} = (1+z_{\text{dec}})T_{\nu,0} \approx (1+1090)\times 1.95\text{ K} = 2127\text{ K} = 0.183\text{ eV}, \tag{3.39}$$
where I have plugged in values from Eq. (3.166) of the textbook. This sets the lower bound of the mass of neutrinos for the neutrinos to becomes non-relativistic. Thus, the smallest neutrino mass that is observable in the CMB is
$$\boxed{m_\nu \gtrsim T_{\nu,\text{dec}} \approx 0.183\text{ eV}}\,. \tag{3.40}$$

4. Using the lower bound on the sum of the neutrino masses from oscillation experiments, we deduce
$$\sum m_\nu > 0.06\text{ eV} \gtrsim T_{\nu,\text{NR}}. \tag{3.41}$$
Again, invoking the consequence of entropy conservation, we have the redshift for the neutrino to become non-relativistic no later than
$$z_{\nu,\text{NR}} = \frac{T_{\nu,\text{NR}}}{T_{\nu,0}} - 1 \lesssim \frac{0.06\text{ eV}}{1.95\text{ K}} - 1 = \frac{0.06\text{ eV}}{1.68\times 10^{-4}\text{ eV}} - 1 \approx \boxed{356}\,. \tag{3.42}$$

5. Note that at the point of neutrino decoupling, neutrino still behaves as relativistic, and the thus, the Eq. (3.65) of the textbook still applies at least up to the point neutrino becomes non-relativistic. However, since neutrino becomes non-relativistic long after neutrino decouples from the thermal bath and it also does not decay, its number is conserved and its number density follows $n_\nu \propto a^{-3}$ from Eq. (2.95) of the textbook. On the other hand, we also know that the photon number density, after the decoupling of the electron-positron annihilation, also follow $n_\gamma \propto a^{-3}$. Thus, the Eq. (3.65) of the textbook still applies today even though neutrinos are massive[1]:

$$n_\nu \approx \frac{3}{4} \times \frac{4}{11} n_\gamma \approx 112\ \text{cm}^{-3} \tag{3.43}$$

per flavor, where I have plugged in $n_{\gamma,0} \approx 410.7\ \text{cm}^{-3}$.

6. However, for the energy density of neutrino, we should use Eq. (3.32) of the textbook:

$$\sum \rho_\nu = \sum \left(m_\nu + \frac{3}{2} T_{\nu,0} \right) n_\nu \approx \sum m_\nu n_\nu, \tag{3.44}$$

where I have approximated the relation by using the fact that $T_{\nu,0} = 1.68 \times 10^{-4}$ eV $\ll \sum m_\nu \sim \mathcal{O}(0.01\ \text{eV})$.

$$\begin{aligned}
\Omega_\nu h^2 &= \frac{\sum \rho_\nu}{\rho_{\gamma,0}} \Omega_\gamma h^2 \\
&\approx \frac{\sum m_\nu n_{\nu,0}}{\rho_{\gamma,0}} \Omega_\gamma h^2 \\
&= \frac{3}{11} \sum m_\nu \frac{n_{\gamma,0}}{\rho_{\gamma,0}} \Omega_\gamma h^2 \\
&\approx \frac{3}{11} \times \frac{410.7\ \text{cm}^{-3}}{0.260\ \text{eV cm}^{-3}} \times 2.473 \times 10^{-5} \sum m_\nu \\
&= \frac{\sum m_\nu}{94\ \text{eV}},
\end{aligned} \tag{3.45}$$

where I have plugged in values from Appendix C.2.3 of the textbook.

From oscillation experiments, $\sum m_\nu > 0.06$ eV, this translates to

$$\Omega_\nu h^2 = \frac{\sum m_\nu}{94\ \text{eV}} > \frac{0.06\ \text{eV}}{94\ \text{eV}} \approx \boxed{6.4 \times 10^{-4}}\,. \tag{3.46}$$

This is not far away from the current cosmological bound $\Omega_\nu h^2 < 0.001$.

7. If a neutrino species is heavy enough to become non-relativistic well before the BBN, it simply behaves like a cold dark matter. This is the essence behind model of sterile neutrinos.

[1] This also justifies why we used $T_{\nu,0} = 1.95$ K in the above derivations.

3.5 Extra relativistic species

1. Recall from Eq. (3.23) of the textbook that for a relativistic species, the energy density is given by

$$\rho_i = \frac{\pi^2}{30} g_i T_i^4 \times \begin{cases} 1 & \text{bosons} \\ \frac{7}{8} & \text{fermions} \end{cases}. \tag{3.47}$$

Thus,

$$\Delta N_{\text{eff}} \equiv \frac{\rho_X}{\rho_\nu} = \frac{g_X T_X^4}{T_\nu^4} \times \begin{cases} \frac{4}{7} & \text{bosons} \\ \frac{1}{2} & \text{fermions} \end{cases}, \tag{3.48}$$

where I have used $g_\nu = 2$. The spin of the new particle s_X enters through the internal degrees of freedom. Assuming that the particle X carries no SM gauge charges (*electrically neutral, colorless, and with no weak isospin*), and that if it is a fermion it is either Majorana-like or possesses only a single chirality,[2] then if massive,

$$g_X = 2s_X + 1 = \begin{cases} 1 & s = 0 \\ 2 & s = \frac{1}{2} \\ 3 & s = 1 \\ 4 & s = \frac{3}{2} \\ 5 & s = 2 \end{cases}, \tag{3.49}$$

and if massless,

$$g_X = \begin{cases} 1 & s = 0 \\ 2 & \text{all other spins} \end{cases}. \tag{3.50}$$

It is well known from QFT that there is no interacting theory for massless particles with $s > 2$ in 4D spacetime. For the above reason, massive particles with $s > 2$, if fundamental, violate the unitarity bound because their scattering amplitudes grow without bound with energy[3]. Therefore, we restrict attention to spins with $s \leq 2$.

From entropy conservation, $g_{*s,i}(aT_i)^3 = \text{const}$, we have

$$\left(\frac{T_X}{T_\nu}\right)^4 = \left(\frac{g_{*s}(T_{\text{dec},\nu})}{g_{*s}(T_{\text{dec},X})}\right)^{\frac{4}{3}}. \tag{3.51}$$

The effective numbers of relativistic species in entropy of X and of neutrino evolve together until X decoupled from the thermal bath, which is why $T_{\text{dec},X}$ comes into play.

Putting everything together,

$$\Delta N_{\text{eff}} \equiv \frac{\rho_X}{\rho_\nu} = g_X \left(\frac{g_{*s}(T_{\text{dec},\nu})}{g_{*s}(T_{\text{dec},X})}\right)^{\frac{4}{3}} \times \begin{cases} \frac{4}{7} & \text{bosons} \\ \frac{1}{2} & \text{fermions} \end{cases}. \tag{3.52}$$

[2] These assumptions are made for a simple expression of g_X. Relaxing any of them would increase g_X multiplicatively. In reality, such a particle would almost certainly decouple from the SM thermal bath well before the EWPT since it interacts only feebly with the rest of the plasma, unless some exotic BSM interaction is introduced. This is exactly what would happen if there were a right-handed neutrino.

[3] This is exactly analogous to the case of the longitudinal polarizations of massive weak gauge bosons, which would violate unitarity bound without the Higgs mechanism. They must possess a well-defined massless limit to be fundamental particles.

Recall that neutrinos decouple shortly before electron-positron annihilation, at which point $g_{*s}(T_{\text{dec},\nu}) = 2 + \frac{7}{8} \times 4 = \frac{11}{2}$. Meanwhile,

$$g_{*s}(T_{\text{dec},X}) = \begin{cases} \frac{247}{4} & \text{shortly before QCDPT} \\ \frac{69}{4} & \text{shortly after QCDPT, but before } \mu,\, \pi \text{ annihilation} \\ \frac{43}{4} & \text{shortly after QCDPT and } \mu,\, \pi \text{ annihilation} \end{cases}, \tag{3.53}$$

we have

$$\Delta N_{\text{eff}} \approx \boxed{g_X \times \begin{cases} \frac{4}{7} & \text{bosons} \\ \frac{1}{2} & \text{fermions} \end{cases} \times \begin{cases} 0.0398 & \text{shortly before QCDPT} \\ 0.218 & \text{shortly after QCDPT, but before } \mu,\, \pi \text{ annihilation} \\ 0.409 & \text{shortly after QCDPT and } \mu,\, \pi \text{ annihilation} \end{cases}}, \tag{3.54}$$

where g_X is given in Eq. (3.49) and Eq. (3.50). The lesson is that the earlier X decouples, the smaller its impact on ΔN_{eff}. This is consistent with the boxed discussion below Table 3.2 of the textbook, which notes that if neutrinos are Dirac particles, half of the degrees of freedom must have decoupled in the very early universe. The Planck constrains $N_{\text{eff}} = 2.99 \pm 0.17$ at 2σ including BAO data [2], while the SM predicts $N_{\text{eff}} = 3.046$. We therefore observe that relativistic species decoupling after the QCD phase transition are essentially ruled out at the 2σ level.

Side Remark: An aside on neutrinos, which the textbook does not explain very clearly, is that there is no issue with them being purely Dirac. The reason is that the right-handed *chiral field* does not couple to any SM gauge interactions. As a "neutr"-ino, it is electrically neutral. As a lepton, it is colorless. And being right-handed, it carries no weak isospin. Since it interacts only with the Higgs boson, whose Yukawa coupling with neutrinos is extremely small (assuming neutrino masses arise via electroweak symmetry breaking as for the other SM fermions), it must decouple from the thermal bath well before even the electroweak phase transition, leaving negligible imprints on N_{eff}.

2. From Eq. (3.64) of the textbook,

$$g_* = 2 + \frac{7}{8} \times 2N_{\text{eff}} \left(\frac{4}{11}\right)^{\frac{4}{3}} = 2 + \frac{7}{8} \times 2 \times (3.046 + \Delta N_{\text{eff}}) \left(\frac{4}{11}\right)^{\frac{4}{3}} \approx \boxed{3.38 + 0.45\Delta N_{\text{eff}}}\,. \tag{3.55}$$

For a radiation-dominated universe, $H \propto \sqrt{\rho} \propto \sqrt{g_*}$. Hence, an increase in g_* increases the Hubble expansion rate H. The discussion then parallels that in Problem 3.3, part 3.

3.6 Gravitinos as dark matter

1. Starting from the general Boltzmann equation (cf. Eq. (3.93) of the textbook), where the RHS is given by the gravitino production interaction rate $\Gamma_g n_g$, we have

$$\frac{1}{a^3}\frac{\mathrm{d}(n_i a^3)}{\mathrm{d}t} = \Gamma_g n_g. \tag{3.56}$$

Let $N_g \equiv \frac{n_g}{s}$, and recall that entropy conservation (cf. Eq. (3.44) of the textbook) gives

$$\frac{\mathrm{d}(sa^3)}{\mathrm{d}t} = 0 \implies \frac{\mathrm{d}s}{\mathrm{d}t} = -3Hs, \tag{3.57}$$

so the Boltzmann equation can be rewritten as

$$\frac{\mathrm{d}N_g}{\mathrm{d}t} = \frac{1}{s}\frac{\mathrm{d}n_g}{\mathrm{d}t} - \frac{n_g}{s^2}\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1}{s}\left(\frac{\mathrm{d}n_g}{\mathrm{d}t} + 3Hn_g\right) = \frac{\Gamma_g n_g}{s}. \tag{3.58}$$

In the radiation-dominated universe, $H \propto T^2$ (cf. Eq. (3.55) of the textbook), and away from particle mass thresholds, $T \propto a^{-1}$ (cf. Eq. (3.54) of the textbook), so

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{\mathrm{d}T}{\mathrm{d}a}\frac{\mathrm{d}a}{\mathrm{d}t} = -HT. \tag{3.59}$$

Hence,

$$\frac{\mathrm{d}N_g}{\mathrm{d}T} = \frac{\mathrm{d}N_g}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}T} = -\frac{\Gamma_g n_g}{HTs}. \tag{3.60}$$

Integrating with respect to the temperature of the current universe and cutting off at the reheating temperature gives

$$N_g \sim \int_0^{T_R} \frac{\mathrm{d}T}{T}\frac{\Gamma_g}{H}\frac{n_g}{s}. \tag{3.61}$$

Note that although we used an approximation valid during the radiation-dominated era, we have integrated from zero temperature. However, since gravitinos are produced most efficiently during reheating after inflation, and since $T_R \gg T_{\mathrm{EWPT}} \sim 100$ GeV $\gg$ $T_{\text{M-R equal}} \sim 0.80$ eV, the lower-temperature regime only alters the result negligibly.

Using $\Gamma_g \sim \frac{T^3}{M_{\mathrm{Pl}}^2}$, $\frac{n_g}{s} \sim \mathcal{O}(1)$, and $H \sim \frac{T^2}{M_{\mathrm{Pl}}}$,

$$N_{g,0} \sim \int_0^{T_R} \frac{\mathrm{d}T}{T}\frac{\Gamma_g}{H}\frac{n_g}{s} \sim \int_0^{T_R} \frac{\mathrm{d}T}{M_{\mathrm{Pl}}} = \boxed{\frac{T_R}{M_{\mathrm{Pl}}}}. \tag{3.62}$$

Side Remark: A simple alternative derivation follows from dimensional analysis. The only relevant scales are M_{Pl} (since the gravitino arises from supergravity, whose natural scale is M_{Pl}) and T_R (the typical energy available to produce gravitinos). The number of particles in a comoving volume is dimensionless, so it must scale as either $\left(\frac{T_R}{M_{\mathrm{Pl}}}\right)^n$ or $\left(\frac{M_{\mathrm{Pl}}}{T_R}\right)^n$. The correct choice has T_R in the numerator: taking $M_{\mathrm{Pl}} \to \infty$ would decouple gravity, so $N_g \to 0$ (similarly, $N_g \to 0$ as $T_R \to 0$). Thus, only $\left(\frac{T_R}{M_{\mathrm{Pl}}}\right)^n$ has the correct scaling. Unless some mechanism forbids the lowest-order term, the $n = 1$ contribution dominates. Therefore, $N_g \sim \frac{T_R}{M_{\mathrm{Pl}}}$.

2. If there is only one massless neutrino species, then the relativistic species today are photons plus one species of neutrino. The present effective number of relativistic species in entropy is

$$g_{*S}(T_0) = 2 + \frac{7}{8} \times 2 \times \left(\frac{4}{11}\right) = \frac{29}{11}. \tag{3.63}$$

The total entropy density today is (cf. Eq. (3.50) of the textbook)

$$s_0 \equiv \frac{2\pi^2}{45} g_{*S}(T_0) T_0^3 \approx 23.41\ \mathrm{K}^3 \times \left(\frac{k_B}{\hbar c}\right)^3 \approx \boxed{1950\ \mathrm{cm}^{-3}}\,, \tag{3.64}$$

where I used the CMB temperature $T_0 = 2.7255$ K (cf. Appendix C.2.3 of the textbook) and included the factor $\left(\frac{k_B}{\hbar c}\right)^3$ to convert to SI units.

3. The effective number of relativistic species is also modified under supergravity. At T_R[4], one must account for both the graviton (massless, since gravity is long range) and the gravitino[5].

   A massless, spin-2 boson (the graviton) has

$$g_{\text{graviton}} = 2. \tag{3.65}$$

   A massive, spin-$\frac{3}{2}$ Majorana fermion (the gravitino) has[6]

$$g_g = 2 \times \frac{3}{2} + 1 = 4. \tag{3.66}$$

   However, the graviton decouples extremely early (typically at $T \sim M_{\mathrm{Pl}}$) since gravity is so feeble[7], and most likely before inflation itself. Its contribution is exponentially diluted during inflation, so it can be neglected.

   Thus,

$$g_{*S}(T_R) = g_{*S}^{\mathrm{SM}}(T_R) + g_{*S}^{g}(T_R) = 106.75 + \frac{7}{8} \times 4 = 112.25. \tag{3.67}$$

   At T_0, one can simply use the result from Eq. (3.63).

   Then,

$$\begin{aligned}
\Omega_g h^2 &\equiv \frac{\rho_{g,0}}{\rho_{\mathrm{crit},0}} h^2 && \text{(cf. Eq. (2.143) of the textbook)}\\
&\approx \frac{m_g n_{g,0}}{3M_{\mathrm{Pl}}^2 H_0^2} h^2 && \text{(cf. Eq. (2.142) and Eq. (3.32) of the textbook)}\\
&= \frac{m_g N_{g,0} T_0^3 \frac{g_{*S(T_0)}}{g_{*S(T_{\mathrm{dec}})}}}{3M_{\mathrm{Pl}}^2 H_0^2} h^2 && \text{(cf. Eq. (3.106) of the textbook)}\\
&= \frac{m_g T_R T_0^3 \frac{g_{*S(T_0)}}{g_{*S(T_{\mathrm{dec}})}}}{3M_{\mathrm{Pl}}^3 H_0^2} h^2 && \text{(cf. Eq. (3.62))}\\
&\sim 0.0659 \left(\frac{T_R}{10^9\ \mathrm{GeV}}\right)\left(\frac{m_g}{1\ \mathrm{GeV}}\right) \frac{g_{*S(T_0)}}{g_{*S(T_{\mathrm{dec}})}},
\end{aligned} \tag{3.68}$$

[4] Assuming all other superpartners of SM particles are already decoupled after reheating.

[5] It is consistent for the graviton to be massless while the gravitino is massive, since SUSY is broken.

[6] There is no extra factor of 2 since the gravitino is Majorana. In fact, gauge symmetry enforces all gauginos to be Majorana in $\mathcal{N} = 1$ SUSY.

[7] In fact, it is so feeble that the gravitino and graviton likely never entered thermal equilibrium after inflation.

where I used $T_0 \approx 0.235$ meV, $M_{\rm Pl} \approx 2.435 \times 10^{18}$ GeV, and $H_0 = 2.133 \times 10^{-33}$ eVh (cf. Appendix C.2.3 of the textbook).

The decoupling temperature of the gravitino is highly model dependent, and so is $g_{*S}(T_{\rm dec})$. A few illustrative examples are:

$$\boxed{\Omega_g h^2 \sim 0.0659\left(\frac{T_R}{10^9 \text{ GeV}}\right)\left(\frac{m_g}{1 \text{ GeV}}\right) \times \begin{cases} 0.0235 & T_{{\rm dec},g} > \mathcal{O}\big(10^3 \text{ GeV}\big) \\ 0.0404 & T_{{\rm dec},g} \gtrsim T_{\rm QCDPT} \\ 0.430 & T_{{\rm dec},g} < T_{\rm BBN} \end{cases}}\,. \tag{3.69}$$

3.7 Baryon asymmetry

- $\boldsymbol{-3\frac{\dot{a}}{a}n}$**:** Dilution of the particle number density $n(t)$ due to the expansion of the universe.
- $\boldsymbol{-n\bar{n}\langle\sigma v\rangle}$**:** Annihilation with the antiparticle counterpart, with thermally averaged cross section $\langle\sigma v\rangle$, where v is the relative speed between the particle and its antiparticle in the system.
- $\boldsymbol{\bar{P}(t)}$**:** The source term that encapsulates any additional contributions to the number density $n(t)$.

1. Taking the CP conjugate of the Boltzmann equation for $n(t)$, we have

$$\frac{\mathrm{d}\bar{n}}{\mathrm{d}t} = -3\frac{\dot{a}}{a}\bar{n} - n\bar{n}\langle\sigma v\rangle + \bar{P}(t). \tag{3.70}$$

By CPT invariance of QFT, the thermally averaged cross section is unchanged, $\langle\overline{\sigma v}\rangle = \langle\sigma v\rangle$. For what follows, we also impose $P(t) = \bar{P}(t)$. Subtracting this from the Boltzmann equation for $n(t)$ gives

$$\begin{aligned} \frac{\mathrm{d}(n-\bar{n})}{\mathrm{d}t} + 3\frac{\dot{a}}{a}(n-\bar{n}) &= 0 \\ \frac{\mathrm{d}[(n-\bar{n})a^3]}{\mathrm{d}t} &= 0. \end{aligned} \tag{3.71}$$

Hence, $(n-\bar{n})a^3$ is a constant.

2. Starting from the Boltzmann equation for $n(t)$,

$$\begin{aligned} \frac{\mathrm{d}n}{\mathrm{d}t} &= -3\frac{\dot{a}}{a}n - n\bar{n}\langle\sigma v\rangle + P(t) \\ \frac{1}{a^3}\frac{\mathrm{d}(na^3)}{\mathrm{d}t} &= -n^2\langle\sigma v\rangle + P(t), \end{aligned} \tag{3.72}$$

where I used the assumption that, although there is no initial particle–antiparticle symmetry, whatever contributes to $n(t)$ through the source term $P(t)$ contributes equally to $\bar{n}(t)$, so $(n-\bar{n})a^3$ remains constant during the evolution and thus $n = \bar{n}$.

At equilibrium, by definition the number density is at an extremum:

$$0 = \frac{1}{a^3}\frac{\mathrm{d}(na^3)}{\mathrm{d}t}\bigg|_{t=t_{\mathrm{eq}}} = -n_{\mathrm{eq}}^2\langle\sigma v\rangle|_{t=t_{\mathrm{eq}}} + P(t_{\mathrm{eq}}). \tag{3.73}$$

Hence, the RHS vanishes at equilibrium:

$$P(t_{\mathrm{eq}}) = n_{\mathrm{eq}}^2\langle\sigma v\rangle|_{t=t_{\mathrm{eq}}}. \tag{3.74}$$

If, in addition, the source term $P(t)$ is constant up to freeze-out (in general, not true[8]), and the thermally averaged cross section $\langle\sigma v\rangle$ is also constant (typically temperature dependent, though thermal effects often do not change final freeze-out results by orders of magnitude), then the Boltzmann equation can be written as

$$\boxed{\frac{1}{a^3}\frac{\mathrm{d}(na^3)}{\mathrm{d}t} = -\langle\sigma v\rangle\left[n^2 - n_{\mathrm{eq}}^2\right]}\,. \tag{3.75}$$

3. The derivation matches the **Riccati equation** treatment in Section 3.2.2 of the textbook, so we quote the result:

$$\boxed{Y^\infty \approx \frac{x_f}{\lambda}}\,, \tag{3.76}$$

where $x_f \equiv \frac{m}{T_f}$ and $\lambda \equiv \frac{\Gamma(m)}{H(m)} = \frac{m^3\langle\sigma v\rangle}{H(m)}$. If decoupling occurs in the early radiation-dominated era, the Hubble parameter from the Friedmann equation (cf. Eq. (3.55) of the textbook) is

$$H(m) \approx \frac{\pi}{\sqrt{90}}\sqrt{g_*(m)}\frac{m^2}{M_{\mathrm{Pl}}}. \tag{3.77}$$

4. A speed-up in the expansion rate $H(m)$ corresponds to a smaller λ. Then, by Eq. (3.76) (cf. Fig. 3.10 of the textbook), <u>the abundance of surviving massive particles is enhanced</u>. Physically, earlier decoupling (set by $\Gamma \sim H$) leaves less time for annihilation with antiparticles, resulting in a larger freeze-out abundance.

5. Although the proton freeze-out temperature is not known a priori, we can assume for protons that $\frac{m_p}{T_f} > 10$, and thermal protons are always non-relativistic throughout cosmic history. The reason is that QCD phase transition occurs at $T_{\mathrm{QCDPT}} \sim 100$ MeV, while $m_p \approx 1$ GeV; before QCDPT the notion of a "proton" does not exist.

   Since most dynamics occurs when the temperature is at least an order of magnitude below the proton mass, the derivation in Section 3.2.2 requires minor adjustments. Suppose the relevant scale is $T \sim T_{\mathrm{QCDPT}}$ rather than $T \sim m_p$. Redefine

$$x \equiv \frac{T_{\mathrm{QCDPT}}}{T}, \tag{3.78}$$

$$\lambda \equiv \frac{\Gamma(T_{\mathrm{QCDPT}})}{H(T_{\mathrm{QCDPT}})} = \frac{T_{\mathrm{QCDPT}}^3\langle\sigma v\rangle}{H(T_{\mathrm{QCDPT}})}. \tag{3.79}$$

[8]One situation where this may hold is the case discussed above Eq. (3.98) in the textbook. It effectively assumes that the sector coupled to the particle in question remains tightly coupled to the thermal plasma, maintaining its equilibrium densities until the particle freezes out.

Then Eq. (3.76) applies with these modified definitions.

Today's $n_{p,0}$ is (cf. Eq. (3.106) of the textbook)

$$n_{p,0} = Y_p^\infty T_0^3 \frac{g_{*S}(T_0)}{g_{*S}(T_{\rm QCDPT})}, \tag{3.80}$$

while today's $n_{\gamma,0}$ is (cf. Eq. (3.24) of the textbook)

$$n_{\gamma,0} = \frac{2\zeta(3)}{\pi^2} T_0^3, \tag{3.81}$$

so

$$\begin{aligned} \frac{n_{p,0}}{n_{\gamma,0}} &= \frac{\pi^2}{2\zeta(3)} Y_p^\infty \frac{g_{*S}(T_0)}{g_{*S}(T_{\rm QCDPT})} \\ &\approx \frac{\pi^2}{2\zeta(3)} x_f \frac{H(T_{\rm QCDPT})}{T_{\rm QCDPT}^3 \langle\sigma v\rangle} \frac{g_{*S}(T_0)}{g_{*S}(T_{\rm QCDPT})} \\ &= \frac{\pi^3}{2\zeta(3)\sqrt{90}} \frac{x_f}{T_{\rm QCDPT}\langle\sigma v\rangle M_{\rm Pl}} \frac{g_{*S}(T_0)}{\sqrt{g_{*S}(T_{\rm QCDPT})}}. \end{aligned} \tag{3.82}$$

Plugging in $\langle\sigma v\rangle \approx 100\ \text{GeV}^{-2}$, $g_{*S}(T_0) = 3.94$, $g_{*S}(T_{\rm QCDPT}) = 17.25$ (QCDPT occurs well before neutrino decoupling), $T_{\rm QCDPT} \approx 100$ MeV, $M_{\rm Pl} = 2.435 \times 10^{18}$ GeV (cf. Appendix C.2.3 of the textbook), and assuming $x_f \sim 10$ (justification: after chiral symmetry breaking the strong sector is described by chiral perturbation theory with pions as force carriers, so protons should decouple around the pion decoupling scale $T_f \sim \mathcal{O}(10\ \text{MeV})$), we find

$$\frac{n_{p,0}}{n_{\gamma,0}} \approx \boxed{5.30 \times 10^{-19} \ll \eta_{\rm obs} \approx 6 \times 10^{-10}}\,. \tag{3.83}$$

Hence, this cannot explain the present proton-to-photon ratio.

This failure indicates that at least one assumption in the derivation is invalid. Reviewing the assumptions:

- $\boldsymbol{P(t) = \bar{P}(t)}$**:** This is unlikely the issue. A violation of C and CP at $T \sim \mathcal{O}(10\ \text{MeV})$ large enough to account for an $\mathcal{O}(10^9)$ effect would probably already have been observed experimentally (e.g., the LHC operates at $E_{\rm CM} = 14$ TeV)[9].
- $\boldsymbol{P(t)}$ **is a constant till proton freeze-out:** This could fail in principle, but quantitatively it is hard to generate the required $\mathcal{O}(10^9)$ discrepancy. The source must balance the annihilation term at equilibrium yet be several orders of magnitude larger during proton decoupling. As above, new physics at this low scale having such large effect would likely have been detected.
- $\boldsymbol{\langle\sigma v\rangle}$ **is a constant:** Same concern as above. Thermal effects are known and can be computed in QFT; at these low scales they typically yield only $\mathcal{O}(1)$ corrections.

[9] Exceptions via clever or exotic model building do exist, typically with sectors extending beyond the SM. Mesogenesis is one such example.

- **$x_f \sim 10$:** From Eq. (3.76), explaining an $\mathcal{O}(10^9)$ discrepancy would require $x_f \equiv \frac{T_{\rm QCDPT}}{T_f} \to 10^{10}$. Since the relevant scale cannot exceed $T_{\rm QCDPT}$, this implies extremely late decoupling, $T_f \sim \mathcal{O}(0.01 \text{ eV})$, near the era of first star formation. With protons non-relativistic, their equilibrium density would remain exponentially suppressed until then, leaving the universe too dilute to support life.
- **Initial particle-antiparticle symmetry:** As Arthur Conan Doyle put it, *"When you have eliminated all which is impossible, then whatever remains, however improbable, must be the truth."* Contrary to the assumption of initial symmetry, there must be a built-in proton–antiproton asymmetry. Thus, baryogenesis must have occurred before proton freeze-out.

3.8 Big Bang nucleosynthesis

1. The decoupling condition is

$$\Gamma(T_{\rm dec}) \approx H(T_{\rm dec}). \tag{3.84}$$

Recall that the weak interaction rate is approximately (cf. Eq. (3.58) of the textbook)

$$\Gamma_\nu(T) \approx G_F^2 T^5, \tag{3.85}$$

and the Hubble rate is (cf. Eq. (3.55) of the textbook)

$$H = \sqrt{\frac{\pi^2 g_*}{90}} \frac{T^2}{M_{\rm Pl}} = \sqrt{\frac{4\pi^3 G g_*}{45}} T^2. \tag{3.86}$$

Approximating $T_f \approx T_{\rm dec}$, we have

$$\boxed{T_f = \left(\frac{4\pi^3 G g_*}{45 G_F^4}\right)^{\frac{1}{6}}}. \tag{3.87}$$

From Eq. (3.120) of the textbook,

$$\left(\frac{n_n}{n_p}\right)_{\rm eq} = e^{-Q/T}. \tag{3.88}$$

The later (earlier) decoupling occurs—that is, the lower (higher) the freeze-out temperature T_f—the longer (shorter) the neutron–proton ratio tracks its equilibrium value, leading to a smaller (larger) ratio. Since neutrons are either incorporated into deuterium (and subsequently helium) or decay, fewer (more) neutrons implies less (more) deuterium and thus a lower (higher) final helium abundance.

In summary: lower/higher T_f $\to$ smaller/larger neutron–proton ratio $\to$ lower/higher final helium abundance.

2. Recall Eq. (3.138) of the textbook,

$$\left(\frac{n_D}{n_p}\right)_{\text{eq}} \approx \eta \left(\frac{T}{m_p}\right)^{3/2} e^{B_D/T}. \tag{3.89}$$

Nucleation at T_{nuc} occurs when this ratio becomes of order one, $\left(\frac{n_D}{n_p}\right)_{\text{eq}} \sim 1$, and is therefore effectively independent of η. Nevertheless, a larger (smaller) baryon-to-photon ratio η shifts T_{nuc} higher (lower), meaning neutrons have less (more) time to decay. From Eq. (3.133) of the textbook,

$$X_n(t) = X_n^{\infty} e^{-t/\tau_n}, \tag{3.90}$$

this leads to a larger (smaller) neutron abundance X_n, which allows more (less) deuterium to form, and consequently more (less) final helium.

> In summary: larger/smaller baryon-to-photon ratio $\eta \to$ more/less deuterium $\to$ larger/smaller final helium abundance.

3. Recall Eq. (3.87) and Eq. (3.88):

- g_*:

  > Larger $g_* \to$ higher $T_f \to$ <u>higher</u> final helium abundance.

- G_F:

  > Smaller $G_F \to$ higher $T_f \to$ <u>higher</u> final helium abundance.

- G:

  > Larger $G \to$ higher $T_f \to$ <u>higher</u> final helium abundance.

- Q:

  > Larger $Q \to$ smaller $\left(\frac{n_n}{n_p}\right)$ at freeze-out $\to$ <u>lower</u> final helium abundance.

- τ_n: From Eq. (3.133) of the textbook,

$$X_n(t) = X_n^{\infty} e^{-t/\tau_n} \tag{3.91}$$

  > Shorter $\tau_n \to$ smaller neutron abundance $X_n \to$ less deuterium formed $\to$ <u>lower</u> final helium abundance.

- μ_ν:

  > As discussed in Problem 3.3, larger $\mu_\nu \to$ <u>higher</u> final helium abundance.

Chapter 4

Cosmological Inflation

Exercise 4.1

Simply taking derivative directly onto the Eq. (4.25) of the textbook:

$$\begin{aligned}\frac{\mathrm{d}\Omega_k}{\mathrm{d}N} &= (1+3w)\Omega_k - \frac{\Omega_{k,i}e^{(1+3w)N}}{\left[(1-\Omega_{k,i})+\Omega_{k,i}e^{(1+3w)N}\right]^2}\Omega_{k,i}(1+3w)e^{(1+3w)N} \\ &= \boxed{(1+3w)\Omega_k(1-\Omega_k)}\ .\end{aligned} \tag{4.1}$$

Exercise 4.2

4.1 Oscillating scalar field

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Appendices

Appendix A

Elements of General Relativity

Exercise A.1

For $\mu = 0$,

$$\partial_\nu F^{0\nu} = \partial_t F^{00} + \partial_i F^{0i} = 0 + \partial_i E^i \equiv \boldsymbol{\nabla} \cdot \mathbf{E} = \rho = J^0. \tag{A.1}$$

For $\mu = i$,

$$\partial_\nu F^{i\nu} = \partial_t F^{i0} + \partial_j F^{ij} = -\partial_t E^i + \epsilon^{ijk}\partial_j B_k \equiv (\boldsymbol{\nabla} \times \mathbf{B})^i - \partial_t \mathbf{E}^i = \mathbf{J}^i, \tag{A.2}$$

Thus,

$$\boxed{\partial_\nu F^{\mu\nu} = J^\mu}\ . \tag{A.3}$$

Exercise A.2

We adopt natural units ($\hbar = c = 1$), so that Eq. (A.28) from the textbook becomes

$$E = \frac{hc}{\lambda} = \frac{2\pi\hbar c}{\lambda} = \frac{2\pi}{\lambda} \tag{A.4}$$

In the lab frame, where the photon is emitted, its four-momentum can be written as

$$P^\mu = (E, p^x, 0, 0), \tag{A.5}$$

with $E = p^x$ for a photon.

Now consider an observer moving with velocity v relative to the lab frame, in a direction making an angle θ with respect to the x-axis. Performing a Lorentz transformation, the photon energy in the observer's frame is

$$E' = \gamma(E - v\cos\theta\ p^x) = \gamma E(1 - v\cos\theta). \tag{A.6}$$

Hence, the photon wavelength in the observer's frame transforms as

$$\frac{2\pi}{\lambda'} = \frac{2\pi\gamma(1 - v\cos\theta)}{\lambda}, \tag{A.7}$$

or equivalently,

$$\boxed{\frac{\lambda'}{\lambda} = \frac{1}{\gamma(1 - v\cos\theta)} = \frac{\sqrt{1 - v^2}}{1 - v\cos\theta}}\ . \tag{A.8}$$

Derivation of Eq. (A.62) (Geodesic Equation)

To derive the **geodesic equation**, Eq. (A.62) of the textbook,

$$\frac{\mathrm{d}^2x}{\mathrm{d}\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = 0 \tag{A.9}$$

from the extremization of the action, Eq. (A.61) of the textbook,

$$S[x^\mu(\lambda)] = -m\int_0^1 d\lambda\sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}, \tag{A.10}$$

We proceed as follows. Consider a small variation of the path: $x^\mu \to x^\mu + \delta x^\mu$. The variation of the action is

$$\begin{aligned}\delta S &\equiv S[x^\mu + \delta x^\mu] - S[x^\mu] \\ &= m\int_0^1 d\lambda\frac{1}{2L}\left(\delta g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} + g_{\mu\nu}\frac{\mathrm{d}\delta x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} + g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}\delta x^\nu}{\mathrm{d}\lambda}\right) \\ &= m\int_0^1 d\lambda\frac{1}{2L}\left(\delta g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} + 2g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}\delta x^\nu}{\mathrm{d}\lambda}\right),\end{aligned} \tag{A.11}$$

where $L^2 = -g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = \left(\frac{\mathrm{d}\tau}{\mathrm{d}\lambda}\right)^2$. Changing variables from λ to τ, we get

$$\begin{aligned}\delta S &= m\int d\tau\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)\frac{1}{2L}\left(\delta g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu L^2 + 2g_{\mu\nu}\dot{x}^\mu\delta\dot{x}^\nu L^2\right) \\ &= m\int d\tau\frac{1}{2}\left(\partial_\alpha g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\delta x^\alpha - 2\frac{d}{d\tau}(g_{\mu\nu}\dot{x}^\mu)\delta x^\nu\right) \\ &= m\int d\tau\left(\frac{1}{2}\partial_\alpha g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\delta x^\alpha - \partial_\alpha g_{\mu\nu}\dot{x}^\alpha\dot{x}^\mu\delta x^\nu - g_{\mu\nu}\ddot{x}^\mu\delta x^\nu\right) \\ &= -m\int d\tau\left(g_{\mu\nu}\ddot{x}^\mu\delta x^\nu - \frac{1}{2}\partial_\alpha g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\delta x^\alpha + \frac{1}{2}\partial_\alpha g_{\mu\nu}\dot{x}^\alpha\dot{x}^\mu\delta x^\nu + \frac{1}{2}\partial_\mu g_{\alpha\nu}\dot{x}^\alpha\dot{x}^\mu\delta x^\nu\right) \\ &= -m\int d\tau\left[g_{\mu\nu}\ddot{x}^\mu + \left(-\frac{1}{2}\partial_\nu g_{\alpha\beta} + \frac{1}{2}\partial_\alpha g_{\beta\nu} + \frac{1}{2}\partial_\beta g_{\alpha\nu}\right)\dot{x}^\alpha\dot{x}^\beta\right]\delta x^\nu \\ &= -m\int d\tau g_{\mu\nu}\left(\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta\right)\delta x^\nu,\end{aligned} \tag{A.12}$$

where $\dot{x}^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$. Since this integral must vanish for arbitrary variations δx^ν, the integrand must vanish identically, yielding the geodesic equation. Note also that the metric tensor $g_{\mu\nu}$, being a geometric property of the manifold, does not *explicitly* depend on the parameter λ.

Exercise A.3

Show that the Christoffel symbol transforms as

$$\begin{aligned}\Gamma'^\mu_{\lambda\nu} &= \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x^\sigma}{\partial x'^\lambda}\frac{\partial x^\eta}{\partial x'^\nu}\Gamma^\rho_{\sigma\eta} + \frac{\partial x'^\mu}{\partial x^\eta}\frac{\partial^2 x^\eta}{\partial x'^\lambda\partial x'^\nu} \\ &= S^\mu{}_\rho(S^{-1})_\lambda{}^\sigma(S^{-1})_\nu{}^\eta\Gamma^\rho_{\sigma\eta} + S^\mu{}_\eta(S^{-1})_\lambda{}^\rho\partial_\rho(S^{-1})_\nu{}^\eta.\end{aligned} \tag{A.13}$$

We begin from the definition of the Christoffel symbols in the primed coordinates:

$$
\begin{aligned}
\Gamma'^{\mu}_{\lambda\nu} &= \frac{1}{2} g'^{\mu\alpha} (\partial'_\lambda g'_{\nu\alpha} + \partial'_\nu g'_{\lambda\alpha} - \partial'_\alpha g'_{\lambda\nu}) \\
&= \frac{1}{2} S^{\mu}{}_{\rho} S^{\alpha}{}_{\beta} g^{\rho\beta} \Big\{ (S^{-1})_{\lambda}{}^{\sigma} \partial_\sigma \left[(S^{-1})_{\nu}{}^{\eta} (S^{-1})_{\alpha}{}^{\gamma} g_{\eta\gamma} \right] + (S^{-1})_{\nu}{}^{\eta} \partial_\eta \left[(S^{-1})_{\lambda}{}^{\sigma} (S^{-1})_{\alpha}{}^{\gamma} g_{\sigma\gamma} \right] \\
&\qquad - (S^{-1})_{\alpha}{}^{\gamma} \partial_\gamma \left[(S^{-1})_{\lambda}{}^{\sigma} (S^{-1})_{\nu}{}^{\eta} g_{\sigma\eta} \right] \Big\}.
\end{aligned}
\tag{A.14}
$$

Focusing on the terms where derivatives act only on the metric tensor (i.e., the tensorial part), we find:

$$
\begin{aligned}
\text{Tensorial part} &= \frac{1}{2} S^{\mu}{}_{\rho} S^{\alpha}{}_{\beta} g^{\rho\beta} \Big\{ (S^{-1})_{\lambda}{}^{\sigma} (S^{-1})_{\nu}{}^{\eta} (S^{-1})_{\alpha}{}^{\gamma} g_{\eta\gamma,\sigma} + (S^{-1})_{\nu}{}^{\eta} (S^{-1})_{\lambda}{}^{\sigma} (S^{-1})_{\alpha}{}^{\gamma} g_{\sigma\gamma,\eta} \\
&\qquad - (S^{-1})_{\alpha}{}^{\gamma} (S^{-1})_{\lambda}{}^{\sigma} (S^{-1})_{\nu}{}^{\eta} g_{\sigma\eta,\gamma} \Big\} \\
&= \frac{1}{2} S^{\mu}{}_{\rho} g^{\rho\beta} \left[(S^{-1})_{\lambda}{}^{\sigma} (S^{-1})_{\nu}{}^{\eta} \delta^{\gamma}_{\beta} g_{\eta\gamma,\sigma} + (S^{-1})_{\nu}{}^{\eta} (S^{-1})_{\lambda}{}^{\sigma} \delta^{\gamma}_{\beta} g_{\sigma\gamma,\eta} - \delta^{\gamma}_{\beta} (S^{-1})_{\lambda}{}^{\sigma} (S^{-1})_{\nu}{}^{\eta} g_{\sigma\eta,\gamma} \right] \\
&= \frac{1}{2} S^{\mu}{}_{\rho} g^{\rho\beta} \left[(S^{-1})_{\lambda}{}^{\sigma} (S^{-1})_{\nu}{}^{\eta} g_{\eta\beta,\sigma} + (S^{-1})_{\nu}{}^{\eta} (S^{-1})_{\lambda}{}^{\sigma} g_{\sigma\beta,\eta} - (S^{-1})_{\lambda}{}^{\sigma} (S^{-1})_{\nu}{}^{\eta} g_{\sigma\eta,\beta} \right] \\
&= S^{\mu}{}_{\rho} (S^{-1})_{\lambda}{}^{\sigma} (S^{-1})_{\nu}{}^{\eta} \frac{1}{2} g^{\rho\beta} (g_{\eta\beta,\sigma} + g_{\sigma\beta,\eta} - g_{\sigma\eta,\beta}) \\
&= S^{\mu}{}_{\rho} (S^{-1})_{\lambda}{}^{\sigma} (S^{-1})_{\nu}{}^{\eta} \Gamma^{\rho}_{\sigma\eta},
\end{aligned}
\tag{A.15}
$$

which confirms the expected transformation law for a rank $(1, 2)$ tensor.

The remaining terms, where derivatives act on the coordinate transformation matrices themselves, form the non-tensorial part:

$$\begin{aligned}
\text{Non-tensorial part} &= \frac{1}{2}S^{\mu}{}_{\rho}S^{\alpha}{}_{\beta}g^{\rho\beta}\Big\{(S^{-1})_{\lambda}{}^{\sigma}\partial_{\sigma}\left[(S^{-1})_{\nu}{}^{\eta}(S^{-1})_{\alpha}{}^{\gamma}\right]g_{\eta\gamma} + (S^{-1})_{\nu}{}^{\eta}\partial_{\eta}\left[(S^{-1})_{\lambda}{}^{\sigma}(S^{-1})_{\alpha}{}^{\gamma}\right]g_{\sigma\gamma} \\
&\qquad - (S^{-1})_{\alpha}{}^{\gamma}\partial_{\gamma}\left[(S^{-1})_{\lambda}{}^{\sigma}(S^{-1})_{\nu}{}^{\eta}\right]g_{\sigma\eta}\Big\} \\
&= \frac{1}{2}S^{\mu}{}_{\rho}S^{\alpha}{}_{\beta}g^{\rho\beta}g_{\eta\gamma}\Big\{(S^{-1})_{\lambda}{}^{\sigma}\partial_{\sigma}\left[(S^{-1})_{\nu}{}^{\eta}(S^{-1})_{\alpha}{}^{\gamma}\right] + (S^{-1})_{\nu}{}^{\sigma}\partial_{\sigma}\left[(S^{-1})_{\lambda}{}^{\eta}(S^{-1})_{\alpha}{}^{\gamma}\right] \\
&\qquad - (S^{-1})_{\alpha}{}^{\sigma}\partial_{\sigma}\left[(S^{-1})_{\nu}{}^{\eta}(S^{-1})_{\lambda}{}^{\gamma}\right]\Big\} \\
&= \frac{1}{2}S^{\mu}{}_{\rho}S^{\alpha}{}_{\beta}g^{\rho\beta}g_{\eta\gamma}\Big[(S^{-1})_{\lambda}{}^{\sigma}(S^{-1})_{\nu,\sigma}{}^{\eta}(S^{-1})_{\alpha}{}^{\gamma} + (S^{-1})_{\lambda}{}^{\sigma}(S^{-1})_{\nu}{}^{\eta}(S^{-1})_{\alpha,\sigma}{}^{\gamma} \\
&\qquad + (S^{-1})_{\nu}{}^{\sigma}(S^{-1})_{\lambda,\sigma}{}^{\eta}(S^{-1})_{\alpha}{}^{\gamma} + (S^{-1})_{\nu}{}^{\sigma}(S^{-1})_{\lambda}{}^{\eta}(S^{-1})_{\alpha,\sigma}{}^{\gamma} \\
&\qquad - (S^{-1})_{\alpha}{}^{\sigma}(S^{-1})_{\nu,\sigma}{}^{\eta}(S^{-1})_{\lambda}{}^{\gamma} - (S^{-1})_{\alpha}{}^{\sigma}(S^{-1})_{\nu}{}^{\eta}(S^{-1})_{\lambda,\sigma}{}^{\gamma}\Big] \\
&= \frac{1}{2}S^{\mu}{}_{\rho}g^{\rho\beta}g_{\eta\gamma}\Big[(S^{-1})_{\lambda}{}^{\sigma}(S^{-1})_{\nu,\sigma}{}^{\eta}\delta^{\gamma}_{\beta} + (S^{-1})_{\nu}{}^{\eta}(S^{-1})_{\lambda,\beta}{}^{\gamma} \\
&\qquad + (S^{-1})_{\nu}{}^{\sigma}(S^{-1})_{\lambda,\sigma}{}^{\eta}\delta^{\gamma}_{\beta} + (S^{-1})_{\lambda}{}^{\eta}(S^{-1})_{\nu,\beta}{}^{\gamma} \\
&\qquad - \delta^{\sigma}_{\beta}(S^{-1})_{\nu,\sigma}{}^{\eta}(S^{-1})_{\lambda}{}^{\gamma} - \delta^{\sigma}_{\beta}(S^{-1})_{\nu}{}^{\eta}(S^{-1})_{\lambda,\sigma}{}^{\gamma}\Big] \\
&= \frac{1}{2}S^{\mu}{}_{\eta}\left[(S^{-1})_{\lambda}{}^{\sigma}(S^{-1})_{\nu,\sigma}{}^{\eta} + (S^{-1})_{\nu}{}^{\sigma}(S^{-1})_{\lambda,\sigma}{}^{\eta}\right] \\
&\quad + \frac{1}{2}S^{\mu}{}_{\rho}g^{\rho\beta}g_{\eta\gamma}\Big[(S^{-1})_{\nu}{}^{\eta}(S^{-1})_{\lambda,\beta}{}^{\gamma} + (S^{-1})_{\lambda}{}^{\eta}(S^{-1})_{\nu,\beta}{}^{\gamma} \\
&\qquad - (S^{-1})_{\nu,\beta}{}^{\eta}(S^{-1})_{\lambda}{}^{\gamma} - (S^{-1})_{\nu}{}^{\eta}(S^{-1})_{\lambda,\beta}{}^{\gamma}\Big] \\
&= \frac{1}{2}S^{\mu}{}_{\eta}\left[(S^{-1})_{\lambda}{}^{\rho}(S^{-1})_{\nu,\rho}{}^{\eta} + (S^{-1})_{\lambda}{}^{\rho}(S^{-1})_{\nu,\rho}{}^{\eta}\right] \\
&= S^{\mu}{}_{\eta}(S^{-1})_{\lambda}{}^{\rho}(S^{-1})_{\nu,\rho}{}^{\eta},
\end{aligned} \tag{A.16}$$

as required. In deriving this, we used the identity

$$S^{\alpha}{}_{\beta}(S^{-1})_{\lambda}{}^{\sigma}(S^{-1})_{\alpha,\sigma}{}^{\gamma} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}}\frac{\partial x^{\sigma}}{\partial x'^{\lambda}}\frac{\partial^2 x^{\gamma}}{\partial x^{\sigma}\partial x'^{\alpha}} = \frac{\partial^2 x^{\gamma}}{\partial x^{\beta}\partial x'^{\lambda}} = (S^{-1})_{\lambda,\beta}{}^{\gamma}, \tag{A.17}$$

as well as the fact that the metric tensor $g_{\eta\gamma}$ is symmetric.

Thus, Eq. (A.93) of the textbook is established:

$$\boxed{\Gamma'^{\mu}_{\lambda\nu} = S^{\mu}{}_{\rho}(S^{-1})_{\lambda}{}^{\sigma}(S^{-1})_{\nu}{}^{\eta}\Gamma^{\rho}_{\sigma\eta} + S^{\mu}{}_{\eta}(S^{-1})_{\lambda}{}^{\rho}(S^{-1})_{\nu,\rho}{}^{\eta}}\,. \tag{A.18}$$

Using the provided hint,

$$(S^{-1})_{\nu,\rho}{}^{\eta}S^{\nu}{}_{\alpha} = -(S^{-1})_{\nu}{}^{\eta}S^{\nu}{}_{\alpha,\rho}, \tag{A.19}$$

and substituting the full transformation into Eq. (A.92) of the textbook:

$$\begin{aligned}
\nabla'_{\lambda}V'^{\mu} &= \partial'_{\lambda}V'^{\mu} + \Gamma'^{\mu}_{\lambda\nu}V'^{\nu} \\
&= (S^{-1})_{\lambda}{}^{\sigma}S^{\mu}{}_{\nu}\partial_{\sigma}V^{\nu} + \left[(S^{-1})_{\lambda}{}^{\sigma}S^{\mu}{}_{\nu,\sigma}\right]V^{\nu} \\
&\qquad + S^{\mu}{}_{\rho}(S^{-1})_{\lambda}{}^{\sigma}(S^{-1})_{\nu}{}^{\eta}\Gamma^{\rho}_{\sigma\eta}S^{\nu}{}_{\alpha}V^{\alpha} + S^{\mu}{}_{\eta}(S^{-1})_{\lambda}{}^{\rho}(S^{-1})_{\nu,\rho}{}^{\eta}S^{\nu}{}_{\alpha}V^{\alpha} \\
&= (S^{-1})_{\lambda}{}^{\sigma}S^{\mu}{}_{\nu}\left(\partial_{\sigma}V^{\nu} + \Gamma^{\nu}_{\sigma\eta}V^{\eta}\right) + \left[(S^{-1})_{\lambda}{}^{\sigma}S^{\mu}{}_{\nu,\sigma}\right]V^{\nu} - S^{\mu}{}_{\eta}(S^{-1})_{\lambda}{}^{\rho}(S^{-1})_{\nu}{}^{\eta}S^{\nu}{}_{\alpha,\rho}V^{\alpha} \\
&= (S^{-1})_{\lambda}{}^{\sigma}S^{\mu}{}_{\nu}\nabla_{\sigma}V^{\nu} + \left[(S^{-1})_{\lambda}{}^{\sigma}S^{\mu}{}_{\nu,\sigma}\right]V^{\nu} - \left[(S^{-1})_{\lambda}{}^{\rho}S^{\mu}{}_{\alpha,\rho}\right]V^{\alpha} \\
&= \boxed{(S^{-1})_{\lambda}{}^{\sigma}S^{\mu}{}_{\nu}\nabla_{\sigma}V^{\nu}}\,,
\end{aligned} \tag{A.20}$$

as expected as a rank $(1,1)$ tensor.

Exercise A.4

We begin by evaluating the covariant derivative of a scalar function defined as the contraction $f = W_{\nu}V^{\nu}$. Using the product rule, we have

$$\nabla_{\mu}f = \nabla_{\mu}(W_{\nu}V^{\nu}) = \partial_{\mu}(W_{\nu}V^{\nu}) = (\partial_{\mu}W_{\nu})V^{\nu} + W_{\nu}(\partial_{\mu}V^{\nu}). \tag{A.21}$$

Alternatively, applying the definition of the covariant derivative directly to the contraction,

$$\nabla_{\mu}f = \nabla_{\mu}(W_{\nu}V^{\nu}) = (\nabla_{\mu}W_{\nu})V^{\nu} + W_{\nu}(\nabla_{\mu}V^{\nu}) = (\nabla_{\mu}W_{\nu})V^{\nu} + W_{\nu}(\partial_{\mu}V^{\nu} + \Gamma^{\nu}_{\mu\alpha}V^{\alpha}). \tag{A.22}$$

Comparing the two expressions, we isolate the covariant derivative of the covector W_{ν} as follows:

$$(\nabla_{\mu}W_{\nu})V^{\nu} = (\partial_{\mu}W_{\nu})V^{\nu} - W_{\nu}\Gamma^{\nu}_{\mu\alpha}V^{\alpha} = (\partial_{\mu}W_{\nu} - W_{\alpha}\Gamma^{\alpha}_{\mu\nu})V^{\nu}. \tag{A.23}$$

Since this relation holds for arbitrary V^{ν}, it follows that

$$\boxed{\nabla_{\mu}W_{\nu} = \partial_{\mu}W_{\nu} - \Gamma^{\alpha}_{\mu\nu}W_{\alpha}}\,. \tag{A.24}$$

Derivation of Eq. (A.111) (Riemann Tensor)

Consider two geodesics separated by an infinitesimal displacement vector B^{μ}. The relative velocity between the geodesics is defined as

$$V^{\mu} \equiv \frac{DB^{\mu}}{D\tau} = U^{\nu}\nabla_{\nu}V^{\mu} = \frac{\mathrm{d}B^{\mu}}{\mathrm{d}\tau} + \Gamma^{\mu}_{\sigma\nu}U^{\nu}B^{\sigma}, \tag{A.25}$$

where $U^{\mu} = \frac{dx^{\mu}}{d\tau}$. The corresponding relative acceleration is given by

$$A^{\mu} \equiv \frac{DV^{\mu}}{D\tau} = U^{\nu}\nabla_{\nu}V^{\mu} = \frac{\mathrm{d}V^{\mu}}{\mathrm{d}\tau} + \Gamma^{\mu}_{\sigma\nu}U^{\nu}V^{\sigma}. \tag{A.26}$$

Substituting Eq. (A.25) into Eq. (A.26), we find

$$\begin{aligned} A^\alpha = \frac{D^2B^\alpha}{D\tau^2} &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\mathrm{d}B^\alpha}{\mathrm{d}\tau} + \Gamma^\alpha_{\beta\gamma}U^\beta B^\gamma\right) + \Gamma^\alpha_{\beta\gamma}U^\beta\left(\frac{\mathrm{d}B^\gamma}{\mathrm{d}\tau} + \Gamma^\gamma_{\delta\epsilon}U^\delta B^\epsilon\right) \\ &= \frac{\mathrm{d}^2B^\alpha}{\mathrm{d}\tau^2} + \frac{\mathrm{d}\Gamma^\alpha_{\beta\gamma}}{\mathrm{d}\tau}U^\beta B^\gamma + \Gamma^\alpha_{\beta\gamma}\frac{\mathrm{d}U^\beta}{\mathrm{d}\tau}B^\gamma + 2\Gamma^\alpha_{\beta\gamma}U^\beta\frac{\mathrm{d}B^\gamma}{\mathrm{d}\tau} + \Gamma^\alpha_{\beta\gamma}U^\beta\Gamma^\gamma_{\delta\epsilon}U^\delta B^\epsilon. \end{aligned} \tag{A.27}$$

Note that

$$\frac{\mathrm{d}\Gamma^\alpha_{\beta\gamma}}{\mathrm{d}\tau} = U^\delta\Gamma^\alpha_{\beta\gamma,\delta}, \tag{A.28}$$

and

$$\frac{\mathrm{d}U^\beta}{\mathrm{d}\tau} = -\Gamma^\beta_{\delta\epsilon}U^\delta U^\epsilon, \tag{A.29}$$

where we have used the fact that U^β satisfies the geodesic equation. Since $(x^\alpha + B^\alpha)$ must also trace out a geodesic, we write out its geodesic equation:

$$\frac{\mathrm{d}^2(x^\alpha + B^\alpha)}{\mathrm{d}\tau^2} = -\Gamma^\alpha_{\beta\gamma}(x^\alpha + B^\alpha)\frac{\mathrm{d}(x^\beta + B^\beta)}{\mathrm{d}\tau}\frac{\mathrm{d}(x^\gamma + B^\gamma)}{\mathrm{d}\tau}, \tag{A.30}$$

and subtract the similar one for x^α, yielding

$$\frac{\mathrm{d}^2B^\alpha}{\mathrm{d}\tau^2} = -\Gamma^\alpha_{\beta\gamma,\delta}B^\delta U^\beta U^\gamma - 2\Gamma^\alpha_{\beta\gamma}\frac{\mathrm{d}B^\gamma}{\mathrm{d}\tau}U^\beta = -\Gamma^\alpha_{\beta\delta,\gamma}B^\gamma U^\beta U^\delta - 2\Gamma^\alpha_{\beta\gamma}\frac{\mathrm{d}B^\gamma}{\mathrm{d}\tau}U^\beta. \tag{A.31}$$

Substituting this result into the expression for A^α, we find

$$\begin{aligned} A^\alpha &= -\Gamma^\alpha_{\beta\delta,\gamma}U^\beta U^\delta B^\gamma + \Gamma^\alpha_{\beta\gamma,\delta}U^\delta U^\beta B^\gamma - \Gamma^\alpha_{\beta\gamma}\Gamma^\beta_{\delta\epsilon}U^\delta U^\epsilon B^\gamma + \Gamma^\alpha_{\beta\gamma}\Gamma^\gamma_{\delta\epsilon}U^\beta U^\delta B^\epsilon \\ &= -\left(\Gamma^\alpha_{\beta\delta,\gamma} - \Gamma^\alpha_{\beta\gamma,\delta} + \Gamma^\alpha_{\epsilon\gamma}\Gamma^\epsilon_{\delta\beta} - \Gamma^\alpha_{\delta\epsilon}\Gamma^\epsilon_{\beta\gamma}\right)U^\beta U^\delta B^\gamma \\ &= -R^\alpha{}_{\beta\gamma\delta}U^\beta U^\delta B^\gamma. \end{aligned} \tag{A.32}$$

In the local inertial frame of a freely falling observer, $U^\mu = (1, 0, 0, 0)$. The relevant component of the Riemann tensor is then $R^\alpha{}_{0\gamma0}$. In the static weak-field limit, all time derivatives vanish, so $\Gamma^\alpha_{0\gamma,0} = 0$. Since the Christoffel symbols arise from derivatives of the metric perturbation $h_{\mu\nu}$ (as $\eta_{\mu\nu}$ is constant), each $\Gamma^\alpha_{\beta\gamma}$ is at least first order in $h_{\mu\nu}$, and we can neglect second-order terms. Thus, the only surviving term for the Riemann tensor $R^\alpha{}_{0\gamma0}$ is $\Gamma^\alpha_{00,\gamma}$. Then,

$$\frac{\mathrm{d}^2B^\alpha}{\mathrm{d}\tau^2} = A^\alpha = -R^\alpha{}_{0\gamma0}B^\gamma = -\Gamma^\alpha_{00,\gamma}B^\gamma = -\Gamma^\alpha_{00,i}B^i = \partial_i\left(\frac{1}{2}\eta^{\alpha j}h_{00,j}\right)B^i. \tag{A.33}$$

Note $\frac{\mathrm{d}^2B^0}{\mathrm{d}\tau^2} = 0$, and $\frac{\mathrm{d}^2B^j}{\mathrm{d}\tau^2} = \frac{1}{2}\partial_i\partial^j h_{00}B^i = -B^i\partial_i\partial^j\Phi$, which corresponds precisely to the Newtonian limit of the geodesic deviation equation.

Proof of Riemann Tensor Identities

- $\boldsymbol{R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma}}$:

  First, note that

$$R_{\mu\nu\rho\sigma} = g_{\mu\lambda}R^\lambda{}_{\nu\rho\sigma} = g_{\mu\lambda}(\Gamma^\lambda_{\nu\sigma,\rho} - \Gamma^\lambda_{\nu\rho,\sigma} + \Gamma^\lambda_{\rho\delta}\Gamma^\delta_{\nu\sigma} - \Gamma^\lambda_{\sigma\delta}\Gamma^\delta_{\nu\rho}) = \Gamma_{\mu\nu\sigma,\rho} - \Gamma_{\mu\nu\rho,\sigma} + \Gamma_{\mu\rho\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma_{\mu\sigma\lambda}\Gamma^\lambda_{\nu\rho}, \tag{A.34}$$

where we introduced the shorthand notation

$$\Gamma_{\mu\nu\sigma} \equiv g_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} = g_{\mu\lambda}\frac{1}{2}g^{\lambda\delta}(g_{\sigma\delta,\nu} + g_{\nu\delta,\sigma} - g_{\nu\sigma,\delta}) = \frac{1}{2}(g_{\sigma\mu,\nu} + g_{\nu\mu,\sigma} - g_{\nu\sigma,\mu}). \tag{A.35}$$

Now, consider Riemann normal coordinates, in which $\Gamma^{\mu}_{\alpha\beta}(p) = 0$ at a point p. In these coordinates, the Riemann tensor reduces to

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= \Gamma_{\mu\nu\sigma,\rho} - \Gamma_{\mu\nu\rho,\sigma} \\ &= \frac{1}{2}(g_{\sigma\mu,\nu\rho} + g_{\nu\mu,\sigma\rho} - g_{\nu\sigma,\mu\rho} - g_{\rho\mu,\nu\sigma} - g_{\nu\mu,\rho\sigma} + g_{\nu\rho,\mu\sigma}) \\ &= \frac{1}{2}(g_{\sigma\mu,\nu\rho} - g_{\nu\sigma,\mu\rho} - g_{\rho\mu,\nu\sigma} + g_{\nu\rho,\mu\sigma}), \end{aligned} \tag{A.36}$$

which is antisymmetric under the exchange $\mu \leftrightarrow \nu$. Therefore, in Riemann normal coordinates,

$$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma}. \tag{A.37}$$

Since this is a tensor identity, it holds in all coordinate systems.

- $\boldsymbol{R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho}}$:

  Eq. (A.36) is also antisymmetric under interchange of $\rho \leftrightarrow \sigma$, and hence

$$R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho}. \tag{A.38}$$

  Since this is a tensor identity, it holds in all coordinate systems.

- $\boldsymbol{R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}}$:

  Eq. (A.36) is symmetric under simultaneous swaps $\mu \leftrightarrow \rho$ and $\nu \leftrightarrow \sigma$, so we also have

$$R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}. \tag{A.39}$$

  Since this is a tensor identity, it holds in all coordinate systems.

- $\boldsymbol{R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} = 0}$:

  Again using Eq. (A.36), we compute

$$\begin{aligned} R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} &= \frac{1}{2}(g_{\sigma\mu,\nu\rho} - g_{\nu\sigma,\mu\rho} - g_{\rho\mu,\nu\sigma} + g_{\nu\rho,\mu\sigma} \\ &\quad + g_{\nu\mu,\rho\sigma} - g_{\rho\nu,\mu\sigma} - g_{\sigma\mu,\rho\nu} + g_{\rho\sigma,\mu\nu} \\ &\quad + g_{\rho\mu,\sigma\nu} - g_{\sigma\rho,\mu\nu} - g_{\nu\mu,\sigma\rho} + g_{\sigma\nu,\mu\rho}) \\ &= 0. \end{aligned} \tag{A.40}$$

  Therefore,

$$R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} = 0. \tag{A.41}$$

  Since this is a tensor identity, it holds in all coordinate systems.

- **Bianchi identity**:

  In Riemann normal coordinates, where the Christoffel symbols vanish and covariant derivatives reduce to partial derivatives, we again use Eq. (A.36) to obtain

$$\begin{aligned} R_{\mu\nu\rho\sigma,\lambda} + R_{\lambda\mu\rho\sigma,\nu} + R_{\nu\lambda\rho\sigma,\mu} &= \frac{1}{2}(g_{\sigma\mu,\nu\rho\lambda} - g_{\nu\sigma,\mu\rho\lambda} - g_{\rho\mu,\nu\sigma\lambda} + g_{\nu\rho,\mu\sigma\lambda} \\ &\quad + g_{\sigma\lambda,\mu\rho\nu} - g_{\mu\sigma,\lambda\rho\nu} - g_{\rho\lambda,\mu\sigma\nu} + g_{\mu\rho,\lambda\sigma\nu} \\ &\quad + g_{\sigma\nu,\lambda\rho\mu} - g_{\lambda\sigma,\nu\rho\mu} - g_{\rho\nu,\lambda\sigma\mu} + g_{\lambda\rho,\nu\sigma\mu}) \\ &= 0. \end{aligned} \tag{A.42}$$

  Since this is a tensor equation, the result holds in all coordinate systems. Therefore, we have the Bianchi identity:

$$\nabla_\lambda R_{\mu\nu\rho\sigma} + \nabla_\nu R_{\lambda\mu\rho\sigma} + \nabla_\mu R_{\nu\lambda\rho\sigma} = 0. \tag{A.43}$$

Proof of $\nabla_\sigma g_{\mu\nu} = 0$

$$\begin{aligned} g_{\mu\nu;\sigma} &= g_{\mu\nu,\sigma} - \Gamma^\alpha_{\sigma\mu} g_{\alpha\nu} - \Gamma^\alpha_{\sigma\nu} g_{\alpha\mu} \\ &= g_{\mu\nu,\sigma} - \frac{1}{2} g^{\alpha\lambda}\left(g_{\mu\lambda,\sigma} + g_{\sigma\lambda,\mu} - g_{\mu\sigma,\lambda}\right) g_{\alpha\nu} - \frac{1}{2} g^{\alpha\lambda}\left(g_{\nu\lambda,\sigma} + g_{\sigma\lambda,\nu} - g_{\nu\sigma,\lambda}\right) g_{\alpha\mu} \\ &= g_{\mu\nu,\sigma} - \frac{1}{2}\left(g_{\mu\nu,\sigma} + g_{\sigma\nu,\mu} - g_{\mu\sigma,\nu}\right) - \frac{1}{2}\left(g_{\nu\mu,\sigma} + g_{\sigma\mu,\nu} - g_{\nu\sigma,\mu}\right) \\ &= 0. \end{aligned} \tag{A.44}$$

Proof of $\nabla^\mu R_{\mu\nu} \neq 0$

The Ricci tensor is defined as

$$R_{\mu\nu} \equiv R^\lambda{}_{\mu\lambda\nu}. \tag{A.45}$$

From the Bianchi identity of the Riemann tensor,

$$\begin{gathered} \nabla_\lambda R_{\mu\nu\rho\sigma} + \nabla_\nu R_{\lambda\mu\rho\sigma} + \nabla_\mu R_{\nu\lambda\rho\sigma} = 0, \\ g^{\sigma\lambda} g^{\mu\rho}\left(\nabla_\lambda R_{\mu\nu\rho\sigma} + \nabla_\nu R_{\lambda\mu\rho\sigma} + \nabla_\mu R_{\nu\lambda\rho\sigma}\right) = 0, \\ \nabla^\sigma R_{\nu\sigma} - g^{\sigma\lambda}\nabla_\nu R_{\lambda\sigma} + \nabla^\rho R_{\nu\rho} = 0, \\ \nabla^\mu R_{\mu\nu} = \frac{1}{2}\nabla_\nu R, \end{gathered} \tag{A.46}$$

where $R = g^{\mu\nu} R_{\mu\nu}$ is the Ricci scalar.

Exercise A.5

$$g^{\mu\nu}\delta R_{\mu\nu} = g^{\mu\nu}\left(\delta\Gamma^\lambda_{\mu\nu,\lambda} - \delta\Gamma^\lambda_{\mu\lambda,\nu} + \delta\Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{\mu\nu} + \Gamma^\lambda_{\lambda\rho}\delta\Gamma^\rho_{\mu\nu} - \delta\Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\rho} - \Gamma^\rho_{\mu\lambda}\delta\Gamma^\lambda_{\nu\rho}\right). \tag{A.47}$$

Suppose we work in Riemann normal coordinates at a point p, so that $\Gamma^{\mu}_{\rho\nu}(p) = 0$. In this coordinate system, the variation becomes

$$\begin{aligned} g^{\mu\nu}\delta R_{\mu\nu} &= g^{\mu\nu}\left(\delta\Gamma^{\lambda}_{\mu\nu,\lambda} - \delta\Gamma^{\lambda}_{\mu\lambda,\nu}\right) \\ &= g^{\rho\nu}\delta\Gamma^{\lambda}_{\rho\nu,\lambda} - g^{\mu\nu}\delta\Gamma^{\lambda}_{\nu\lambda,\mu} \\ &= \partial_{\mu}(g^{\rho\nu}\delta\Gamma^{\mu}_{\rho\nu} - g^{\mu\nu}\delta\Gamma^{\lambda}_{\nu\lambda}) \\ &= \partial_{\mu}X^{\mu}, \end{aligned} \tag{A.48}$$

where on the second line, we relabel the dummy indices in the first term as $\mu \to \rho$, and in the second term as $\mu \leftrightarrow \nu$. On the third line, we again relabel the dummy index $\lambda \to \mu$ in the first term. We also used the fact that in normal coordinates, $\Gamma^{\mu}_{\rho\nu}(p) = 0$ implies $g^{\rho\nu}_{;\mu} = 0 \Rightarrow g^{\rho\nu}_{,\mu} = 0$ at the point p. Finally, when we replace the partial derivative with the covariant derivative, this becomes a tensor identity, and thus it holds in all coordinate systems. Therefore, we conclude:

$$g^{\mu\nu}\delta R_{\mu\nu} = \nabla_{\mu}X^{\mu}, \tag{A.49}$$

where

$$X^{\mu} = g^{\rho\nu}\delta\Gamma^{\mu}_{\rho\nu} - g^{\mu\nu}\delta\Gamma^{\lambda}_{\nu\lambda}. \tag{A.50}$$

Proof of $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$

$$\begin{aligned} \delta\sqrt{-g} &\equiv \delta\sqrt{-\det g_{\mu\nu}} \\ &= -\frac{1}{2}(\sqrt{-g})^{-1}\,\delta\big(\det g_{\mu\nu}\big) \\ &= -\frac{1}{2}(\sqrt{-g})^{-1}\,\delta\left(e^{\log\left(\det g_{\mu\nu}\right)}\right) \\ &= -\frac{1}{2}(\sqrt{-g})^{-1}\,\delta\left(e^{-\log\left(\det\left((g_{\mu\nu})^{-1}\right)\right)}\right) \\ &= -\frac{1}{2}(\sqrt{-g})^{-1}\,\delta\left(e^{-\log\left(\det\left(g^{\mu\nu}\right)\right)}\right) \\ &= -\frac{1}{2}(\sqrt{-g})^{-1}\,\delta\left(e^{-\mathrm{Tr}\left(\log\left(g^{\mu\nu}\right)\right)}\right) \\ &= -\frac{1}{2}(\sqrt{-g})^{-1}(-g)\,\delta(\mathrm{Tr}(\log(g^{\mu\nu}))) \\ &= -\frac{1}{2}\sqrt{-g}\,\mathrm{Tr}(\delta(\log(g^{\mu\nu}))) \\ &= -\frac{1}{2}\sqrt{-g}(g^{\mu\nu})^{-1}\delta g^{\mu\nu} \\ &= \boxed{-\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}}\,. \end{aligned} \tag{A.51}$$

Proof that the Minkowski metric is a solution to the Einstein equations

From Eq. (A.156) of the textbook, the Minkowski metric is given by

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}. \tag{A.52}$$

Since all components of the metric tensor are constants, all derivatives vanish, implying that the Christoffel symbols, which depend on derivatives of the metric, vanish identically:

$$\Gamma^{\mu}_{\alpha\beta} = 0. \tag{A.53}$$

As a result, the Riemann curvature tensor, which depends on derivatives and products of the Christoffel symbols, also vanishes. Therefore, the Ricci tensor and the Ricci scalar vanish as well:

$$R_{\mu\nu} = 0, \qquad R = 0. \tag{A.54}$$

Consequently, the Einstein tensor reduces to

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0. \tag{A.55}$$

Hence, the Minkowski metric is indeed a solution of the vacuum ($T_{\mu\nu} = 0$) Einstein equations.

Proof that the Schwarzschild metric is a solution to Einstein equations

From Eq. (A.157) of the textbook, the Schwarzschild metric is given by

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{2GM}{r}\right) & & & \\ & \left(1 - \frac{2GM}{r}\right)^{-1} & & \\ & & r^2 & \\ & & & r^2 \sin^2\theta \end{pmatrix}. \tag{A.56}$$

Since the metric is diagonal, any Christoffel symbol $\Gamma^{\mu}_{\alpha\beta}$ with $\mu \neq \alpha \neq \beta$ vanishes identically. The non-vanishing Christoffel symbols are:

$$\begin{aligned}
\Gamma^{t}_{tr} &= \Gamma^{t}_{rt} = -\Gamma^{r}_{rr} = -\frac{GM}{2GMr - r^2}, \\
\Gamma^{r}_{tt} &= \frac{GM(-2GM + r)}{r^3}, \\
\Gamma^{r}_{\theta\theta} &= 2GM - r, \\
\Gamma^{r}_{\phi\phi} &= (2GM - r)\sin^2\theta, \\
\Gamma^{\theta}_{r\theta} &= \Gamma^{\theta}_{\theta r} = \Gamma^{\phi}_{r\phi} = \Gamma^{\phi}_{\phi r} = \frac{1}{r}, \\
\Gamma^{\theta}_{\phi\phi} &= -\cos\theta\sin\theta, \\
\Gamma^{\phi}_{\theta\phi} &= \Gamma^{\phi}_{\phi\theta} = \cot\theta.
\end{aligned} \tag{A.57}$$

By spherical symmetry, components of the Ricci tensor that mix angular and radial or time coordinates, which imply a preferred angular direction, vanish:

$$R_{t\theta} = R_{t\phi} = R_{r\theta} = R_{r\phi} = R_{\theta\phi} = 0, \tag{A.58}$$

along with their symmetric counterparts.

Moreover, since no component of the metric depends on time, the spacetime exhibits time translation invariance. Hence,

$$R_{rt} = R_{tr} = 0. \tag{A.59}$$

The potentially non-zero components of the Ricci tensor are therefore R_{tt}, R_{rr}, $R_{\theta\theta}$, and $R_{\phi\phi}$. We compute them explicitly:

- $\boldsymbol{R_{tt}}$:

$$\begin{aligned}
R_{tt} &= \Gamma^{\lambda}_{tt,\lambda} - \Gamma^{\lambda}_{t\lambda,t} + \Gamma^{\lambda}_{\lambda\rho}\Gamma^{\rho}_{tt} - \Gamma^{\rho}_{t\lambda}\Gamma^{\lambda}_{t\rho} \\
&= \Gamma^{r}_{tt,r} + (\Gamma^{t}_{tr} + \Gamma^{r}_{rr} + \Gamma^{\theta}_{\theta r} + \Gamma^{\phi}_{\phi r})\Gamma^{r}_{tt} - \Gamma^{t}_{tr}\Gamma^{r}_{tt} - \Gamma^{r}_{tt}\Gamma^{t}_{tr} \\
&= \Gamma^{r}_{tt,r} + (2\Gamma^{r}_{rr} + \Gamma^{\theta}_{\theta r} + \Gamma^{\phi}_{\phi r})\Gamma^{r}_{tt} \\
&= 2GM\left(\frac{3GM - r}{r^4}\right) - 2GM\left(\frac{3GM - r}{r^4}\right) \\
&= 0,
\end{aligned} \tag{A.60}$$

- $\boldsymbol{R_{rr}}$:

$$\begin{aligned}
R_{rr} &= \Gamma^{\lambda}_{rr,\lambda} - \Gamma^{\lambda}_{r\lambda,r} + \Gamma^{\lambda}_{\lambda\rho}\Gamma^{\rho}_{rr} - \Gamma^{\rho}_{r\lambda}\Gamma^{\lambda}_{r\rho} \\
&= \Gamma^{r}_{rr,r} - \Gamma^{t}_{rt,r} - \Gamma^{r}_{rr,r} - \Gamma^{\theta}_{r\theta,r} - \Gamma^{\phi}_{r\phi,r} + (\Gamma^{t}_{tr} + \Gamma^{r}_{rr} + \Gamma^{\theta}_{\theta r} + \Gamma^{\phi}_{\phi r})\Gamma^{r}_{rr} \\
&\quad - (\Gamma^{t}_{rt})^2 - (\Gamma^{r}_{rr})^2 - (\Gamma^{\theta}_{r\theta})^2 - (\Gamma^{\phi}_{r\phi})^2 \\
&= -2GM\frac{GM - r}{r^2(2GM - r)^2} + \frac{2GM}{r^2(2GM - r)} - \frac{2(GM)^2}{r^2(2GM - r)^2} \\
&= 0,
\end{aligned} \tag{A.61}$$

- $\boldsymbol{R_{\theta\theta}}$:

$$\begin{aligned}
R_{\theta\theta} &= \Gamma^{\lambda}_{\theta\theta,\lambda} - \Gamma^{\lambda}_{\theta\lambda,\theta} + \Gamma^{\lambda}_{\lambda\rho}\Gamma^{\rho}_{\theta\theta} - \Gamma^{\rho}_{\theta\lambda}\Gamma^{\lambda}_{\theta\rho} \\
&= \Gamma^{r}_{\theta\theta,r} - \Gamma^{\phi}_{\theta\phi,\theta} + (\Gamma^{t}_{tr} + \Gamma^{r}_{rr} + \Gamma^{\theta}_{\theta r} + \Gamma^{\phi}_{\phi r})\Gamma^{r}_{\theta\theta} - \Gamma^{r}_{\theta\theta}\Gamma^{\theta}_{\theta r} - \Gamma^{\theta}_{\theta r}\Gamma^{r}_{\theta\theta} - (\Gamma^{\phi}_{\theta\phi})^2 \\
&= -1 + \frac{1}{\sin^2\theta} + \frac{2}{r}(2GM - r) - \frac{2}{r}(2GM - r) - \cot^2\theta \\
&= 0,
\end{aligned} \tag{A.62}$$

- $\boldsymbol{R_{\phi\phi}}$:

$$\begin{aligned}
R_{\phi\phi} &= \Gamma^{\lambda}_{\phi\phi,\lambda} - \Gamma^{\lambda}_{\phi\lambda,\phi} + \Gamma^{\lambda}_{\lambda\rho}\Gamma^{\rho}_{\phi\phi} - \Gamma^{\rho}_{\phi\lambda}\Gamma^{\lambda}_{\phi\rho} \\
&= \Gamma^{r}_{\phi\phi,r} + \Gamma^{\theta}_{\phi\phi,\theta} + (\Gamma^{t}_{tr} + \Gamma^{r}_{rr} + \Gamma^{\theta}_{\theta r} + \Gamma^{\phi}_{\phi r})\Gamma^{r}_{\phi\phi} + \Gamma^{\phi}_{\phi\theta}\Gamma^{\theta}_{\phi\phi} - \Gamma^{r}_{\phi\phi}\Gamma^{\phi}_{\phi r} - \Gamma^{\theta}_{\phi\phi}\Gamma^{\phi}_{\phi\theta} - \Gamma^{\phi}_{\phi r}\Gamma^{r}_{\phi\phi} - \Gamma^{\phi}_{\phi\theta}\Gamma^{\theta}_{\phi\phi} \\
&= \Gamma^{r}_{\phi\phi,r} + \Gamma^{\theta}_{\phi\phi,\theta} - \Gamma^{\phi}_{\phi\theta}\Gamma^{\theta}_{\phi\phi} \\
&= -\sin^2\theta - \cos^2\theta + \sin^2\theta + \cot\theta\cos\theta\sin\theta \\
&= 0.
\end{aligned} \tag{A.63}$$

Since the Ricci tensor $R_{\mu\nu}$ is symmetric, it has 10 independent components. We have explicitly computed all independent components and verified that each vanishes. Therefore, the Ricci scalar also vanishes:

$$R = g^{\mu\nu} R_{\mu\nu} = 0. \tag{A.64}$$

Thus, the Schwarzschild metric indeed satisfies the vacuum ($T_{\mu\nu} = 0$) Einstein field equations:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0. \tag{A.65}$$

Proof that the de Sitter metric is a solution to Einstein equations

From Eq. (A.158) of the textbook, the de Sitter metric is given by

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{r^2}{R^2}\right) & & & \\ & \left(1 - \frac{r^2}{R^2}\right)^{-1} & & \\ & & r^2 & \\ & & & r^2 \sin^2\theta \end{pmatrix}. \tag{A.66}$$

Since the metric is diagonal, any Christoffel symbol $\Gamma^{\mu}_{\alpha\beta}$ with $\mu \neq \alpha \neq \beta$ vanishes identically. The non-vanishing Christoffel symbols are:

$$\begin{aligned}
\Gamma^t_{tr} &= \Gamma^t_{rt} = -\Gamma^r_{rr} = \frac{r}{r^2 - R^2}, \\
\Gamma^r_{tt} &= \frac{r^3 - rR^2}{R^4}, \\
\Gamma^r_{\theta\theta} &= -r + \frac{r^3}{R^2}, \\
\Gamma^r_{\phi\phi} &= \frac{r(r^2 - R^2)\sin^2\theta}{R^2}, \\
\Gamma^\theta_{r\theta} &= \Gamma^\theta_{\theta r} = \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r}, \\
\Gamma^\theta_{\phi\phi} &= -\cos\theta\sin\theta, \\
\Gamma^\phi_{\theta\phi} &= \Gamma^\phi_{\phi\theta} = \cot\theta.
\end{aligned} \tag{A.67}$$

Again, like the case of the Schwarzschild metric, spherical symmetry and time translation invariance imply:

$$R_{tr} = R_{t\theta} = R_{t\phi} = R_{r\theta} = R_{r\phi} = R_{\theta\phi} = 0, \tag{A.68}$$

as well as their symmetric counterparts. The only potentially non-vanishing components of the Ricci tensor are R_{tt}, R_{rr}, $R_{\theta\theta}$, and $R_{\phi\phi}$. The structure of the computation mirrors the Schwarzschild case, as both metrics are time-independent, spherically symmetric, diagonal, and radially dependent.

- $\boldsymbol{R_{tt}}$:

$$\begin{aligned} R_{tt} &= \Gamma^{\lambda}_{tt,\lambda} - \Gamma^{\lambda}_{t\lambda,t} + \Gamma^{\lambda}_{\lambda\rho}\Gamma^{\rho}_{tt} - \Gamma^{\rho}_{t\lambda}\Gamma^{\lambda}_{t\rho} \\ &= \Gamma^{r}_{tt,r} + (\Gamma^{t}_{tr} + \Gamma^{r}_{rr} + \Gamma^{\theta}_{\theta r} + \Gamma^{\phi}_{\phi r})\Gamma^{r}_{tt} - \Gamma^{t}_{tr}\Gamma^{r}_{tt} - \Gamma^{r}_{tt}\Gamma^{t}_{tr} \\ &= \Gamma^{r}_{tt,r} + (2\Gamma^{r}_{rr} + \Gamma^{\theta}_{\theta r} + \Gamma^{\phi}_{\phi r})\Gamma^{r}_{tt} \\ &= \frac{3r^2 - R^2}{R^4} + \left(-\frac{2r}{r^2 - R^2} + \frac{2}{r}\right)\frac{r^3 - rR^2}{R^4} \\ &= \frac{3(r^2 - R^2)}{R^4}, \end{aligned} \tag{A.69}$$

- $\boldsymbol{R_{rr}}$:

$$\begin{aligned} R_{rr} &= \Gamma^{\lambda}_{rr,\lambda} - \Gamma^{\lambda}_{r\lambda,r} + \Gamma^{\lambda}_{\lambda\rho}\Gamma^{\rho}_{rr} - \Gamma^{\rho}_{r\lambda}\Gamma^{\lambda}_{r\rho} \\ &= \Gamma^{r}_{rr,r} - \Gamma^{t}_{rt,r} - \Gamma^{r}_{rr,r} - \Gamma^{\theta}_{r\theta,r} - \Gamma^{\phi}_{r\phi,r} + (\Gamma^{t}_{tr} + \Gamma^{r}_{rr} + \Gamma^{\theta}_{\theta r} + \Gamma^{\phi}_{\phi r})\Gamma^{r}_{rr} \\ &\quad - (\Gamma^{t}_{rt})^2 - (\Gamma^{r}_{rr})^2 - (\Gamma^{\theta}_{r\theta})^2 - (\Gamma^{\phi}_{r\phi})^2 \\ &= \left(-\frac{1}{r^2 - R^2} + \frac{2r^2}{(r^2 - R^2)^2}\right) + \frac{2}{r}\left(\frac{-r}{r^2 - R^2}\right) - 2\left(\frac{r}{r^2 - R^2}\right)^2 \\ &= -\frac{3}{r^2 - R^2}, \end{aligned} \tag{A.70}$$

- $\boldsymbol{R_{\theta\theta}}$:

$$\begin{aligned} R_{\theta\theta} &= \Gamma^{\lambda}_{\theta\theta,\lambda} - \Gamma^{\lambda}_{\theta\lambda,\theta} + \Gamma^{\lambda}_{\lambda\rho}\Gamma^{\rho}_{\theta\theta} - \Gamma^{\rho}_{\theta\lambda}\Gamma^{\lambda}_{\theta\rho} \\ &= \Gamma^{r}_{\theta\theta,r} - \Gamma^{\phi}_{\theta\phi,\theta} + (\Gamma^{t}_{tr} + \Gamma^{r}_{rr} + \Gamma^{\theta}_{\theta r} + \Gamma^{\phi}_{\phi r})\Gamma^{r}_{\theta\theta} - \Gamma^{r}_{\theta\theta}\Gamma^{\theta}_{\theta r} - \Gamma^{\theta}_{\theta r}\Gamma^{r}_{\theta\theta} - (\Gamma^{\phi}_{\theta\phi})^2 \\ &= -1 + \frac{3r^2}{R^2} + \frac{1}{\sin^2\theta} - \cot^2\theta \\ &= \frac{3r^2}{R^2}, \end{aligned} \tag{A.71}$$

- $\boldsymbol{R_{\phi\phi}}$:

$$\begin{aligned} R_{\phi\phi} &= \Gamma^{\lambda}_{\phi\phi,\lambda} - \Gamma^{\lambda}_{\phi\lambda,\phi} + \Gamma^{\lambda}_{\lambda\rho}\Gamma^{\rho}_{\phi\phi} - \Gamma^{\rho}_{\phi\lambda}\Gamma^{\lambda}_{\phi\rho} \\ &= \Gamma^{r}_{\phi\phi,r} + \Gamma^{\theta}_{\phi\phi,\theta} - \Gamma^{\phi}_{\phi\theta}\Gamma^{\theta}_{\phi\phi} \\ &= \frac{3r^2\sin^2\theta}{R^2} - \sin^2\theta - \cos^2\theta + \sin^2\theta + \cot\theta\cos\theta\sin\theta \\ &= \frac{3r^2\sin^2\theta}{R^2}. \end{aligned} \tag{A.72}$$

The Ricci scalar R is then

$$R = g^{\mu\nu}R_{\mu\nu} = g^{tt}R_{tt} + g^{rr}R_{rr} + g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi} = \frac{3}{R^2} + \frac{3}{R^2} + \frac{3}{R^2} + \frac{3}{R^2} = \frac{12}{R^2} \tag{A.73}$$

The Einstein tensor $G_{\mu\nu}$ is hence also diagonal, and its non-vanishing components are:

- $\boldsymbol{G_{tt}}$:

$$G_{tt} \equiv R_{tt} - \frac{1}{2}g_{tt}R = \frac{3(r^2 - R^2)}{R^4} - \frac{6}{R^2}\left(\frac{r^2 - R^2}{R^2}\right) = -\frac{3(r^2 - R^2)}{R^4} = \Lambda\frac{r^2 - R^2}{R^2} = -\Lambda g_{tt}, \tag{A.74}$$

- $\boldsymbol{G_{rr}}$:

$$G_{rr} \equiv R_{rr} - \frac{1}{2}g_{rr}R = -\frac{3}{r^2 - R^2} - \frac{6}{R^2}\left(\frac{R^2}{R^2 - r^2}\right) = \frac{3}{r^2 - R^2} = \Lambda\frac{R^2}{r^2 - R^2} = -\Lambda g_{rr}, \tag{A.75}$$

- $\boldsymbol{G_{\theta\theta}}$:

$$G_{\theta\theta} \equiv R_{\theta\theta} - \frac{1}{2}g_{\theta\theta}R = \frac{3r^2}{R^2} - \frac{6}{R^2}r^2 = -\frac{3r^2}{R^2} = \Lambda r^2 = -\Lambda g_{\theta\theta}, \tag{A.76}$$

- $\boldsymbol{G_{\phi\phi}}$:

$$G_{\phi\phi} \equiv R_{\phi\phi} - \frac{1}{2}g_{\phi\phi}R = \frac{3r^2\sin^2\theta}{R^2} - \frac{6}{R^2}r^2\sin^2\theta = -\frac{3r^2\sin^2\theta}{R^2} = \Lambda r^2\sin^2\theta = -\Lambda g_{\phi\phi}, \tag{A.77}$$

where we used the definition $R^2 \equiv 3/\Lambda$.

Hence,

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} \tag{A.78}$$

Therefore, the de Sitter metric indeed solves the vacuum ($T_{\mu\nu} = 0$) Einstein equation with a positive cosmological constant $\Lambda > 0$:

Proof that the anti-de Sitter metric is a solution to Einstein equations

The calculation is identical to the case of the de Sitter metric. One can simply perform the replacement $R^2 \to -R^2$. With this substitution, all intermediate steps and final expressions carry over directly, and the entire derivation remains unchanged.

Bibliography

[1] D. Baumann, Cosmology, Cambridge University Press, 2022. doi:10.1017/9781108937092.

[2] N. Aghanim, et al., Planck 2018 results. VI. Cosmological parameters, Astron. Astrophys. 641 (2020) A6, [Erratum: Astron.Astrophys. 652, C4 (2021)]. arXiv:1807.06209, doi:10.1051/0004-6361/201833910.